Recursion

CS 121: Data Structures

START RECORDING

Attendance Quiz: I/O and Functions

- Scan the QR code, or find today's attendance quiz under the "Quizzes" tab on Canvas
- Password: to be announced in class
- After five minutes, we will discuss the answers

Attendance Quiz: I/O and Functions

- Write your name
- Translate the following pseudocode into a Java program, Bouncer.java

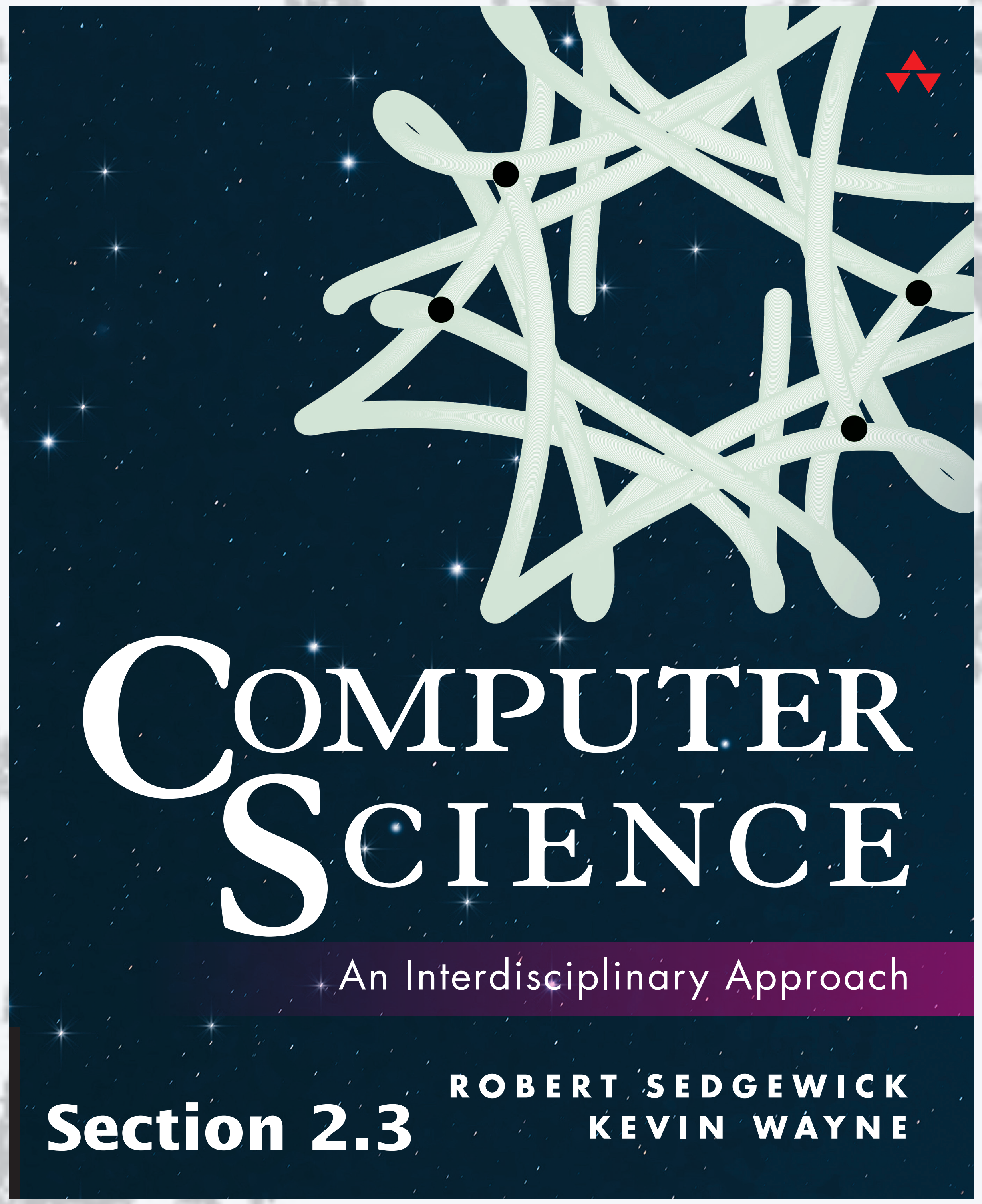
```
The bouncer should ask the user for their age. “What is your age? ”

The bouncer should use a rules() method to check whether the age meets the criteria for
entry into the establishment. Based on the rules, the appropriate answer should be
printed.

 - Age less than 10: “Where are your parents?”
 - Age less than 21: “Sorry, you can’t enter.”
 - Age at least 21: “Welcome!”
```

Outline

- Attendance quiz
- Foundations
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

COMPUTER SCIENCE
SEDGEWICK / WAYNE

PART I: PROGRAMMING IN JAVA



http://introcs.cs.princeton.edu

6. Recursion

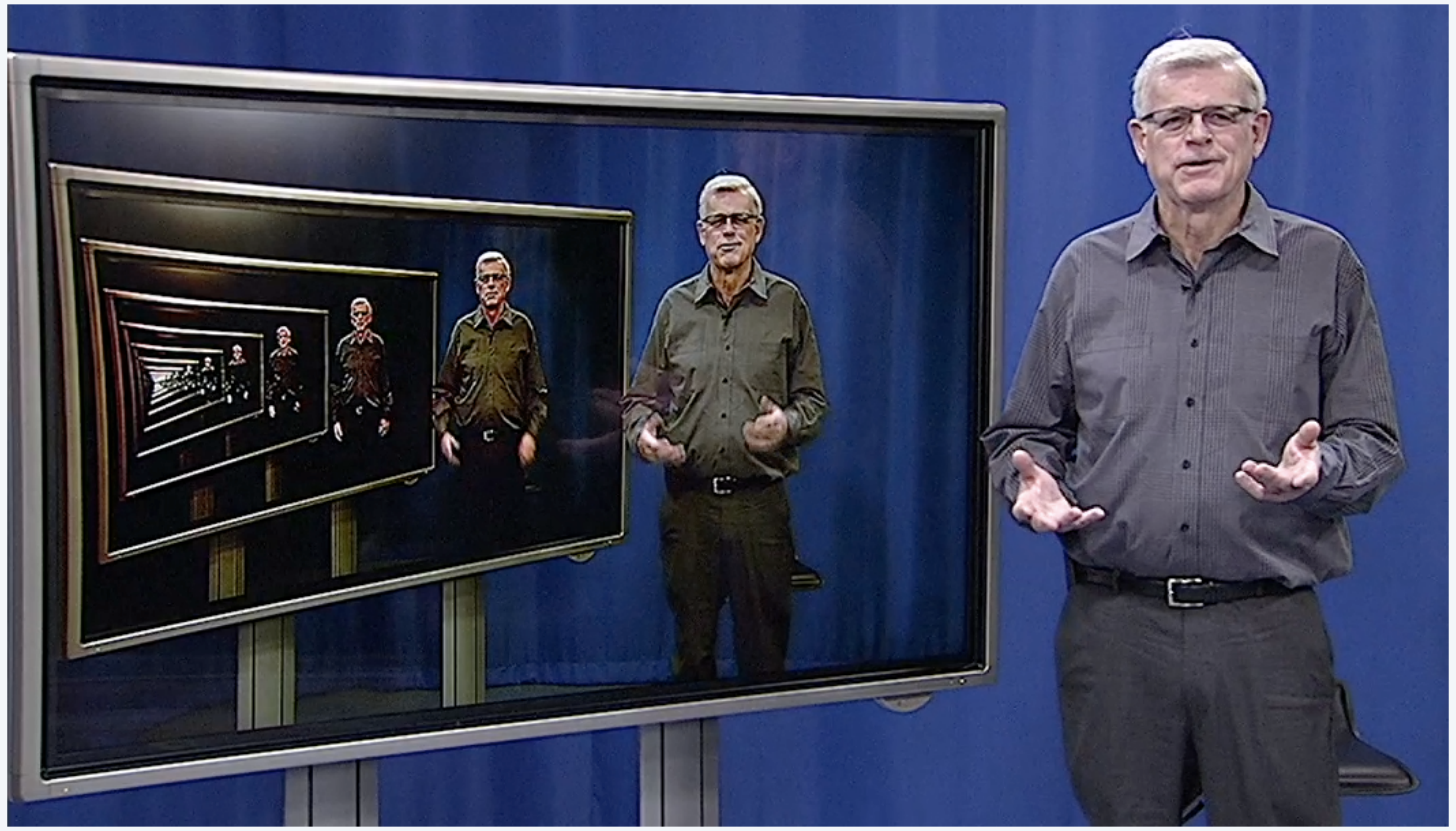
6. Recursion

- **Foundations**
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

Overview

Q. What is recursion?

A. When something is specified in terms of *itself*.

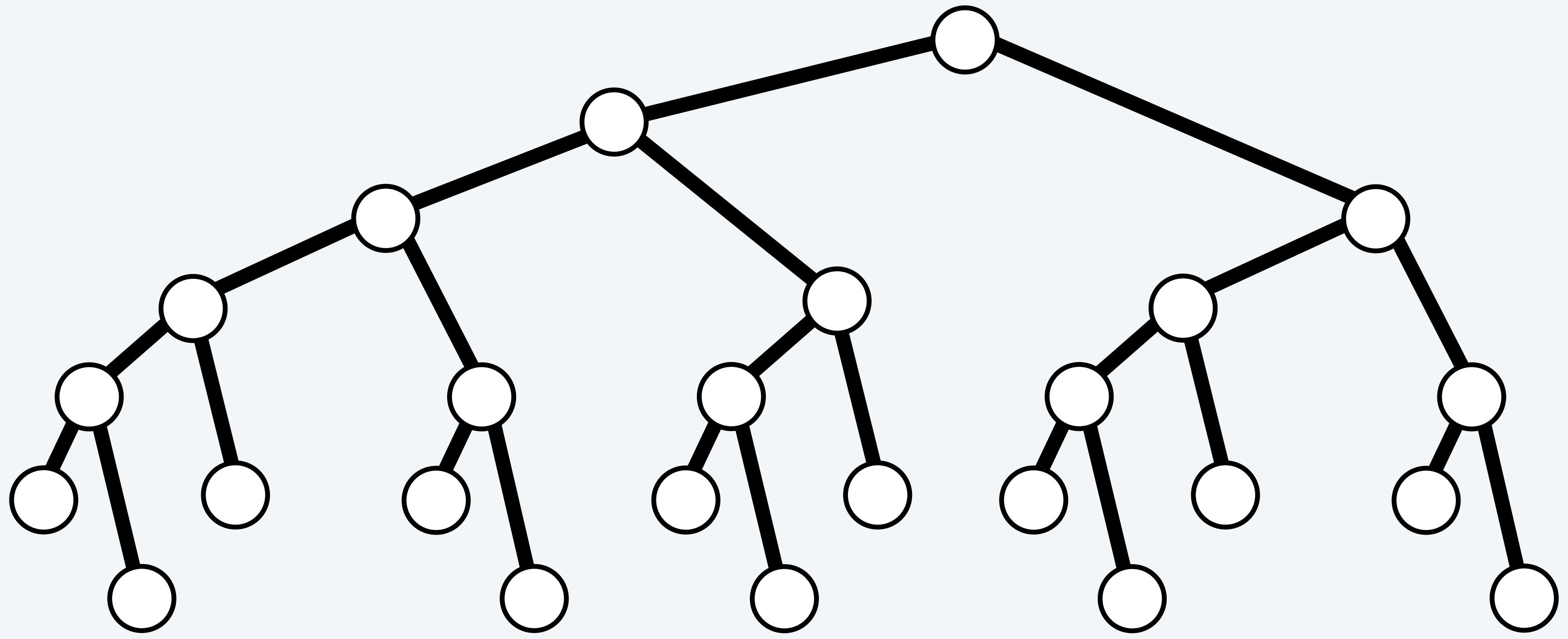


Why learn recursion?

- Represents a new mode of thinking.
- Provides a powerful programming paradigm.
- Enables reasoning about correctness.
- Gives insight into the nature of computation.

Many computational artifacts are *naturally* self-referential.

- File system with folders containing folders.
- Fractal graphical patterns.
- Divide-and-conquer algorithms (stay tuned).



Mathematical induction (quick review)

To prove a statement involving a positive integer N

- Base case. Prove it for some specific values of N.
- Induction step. Assuming that the statement is true for all positive integers less than N, use that fact to prove it for N.

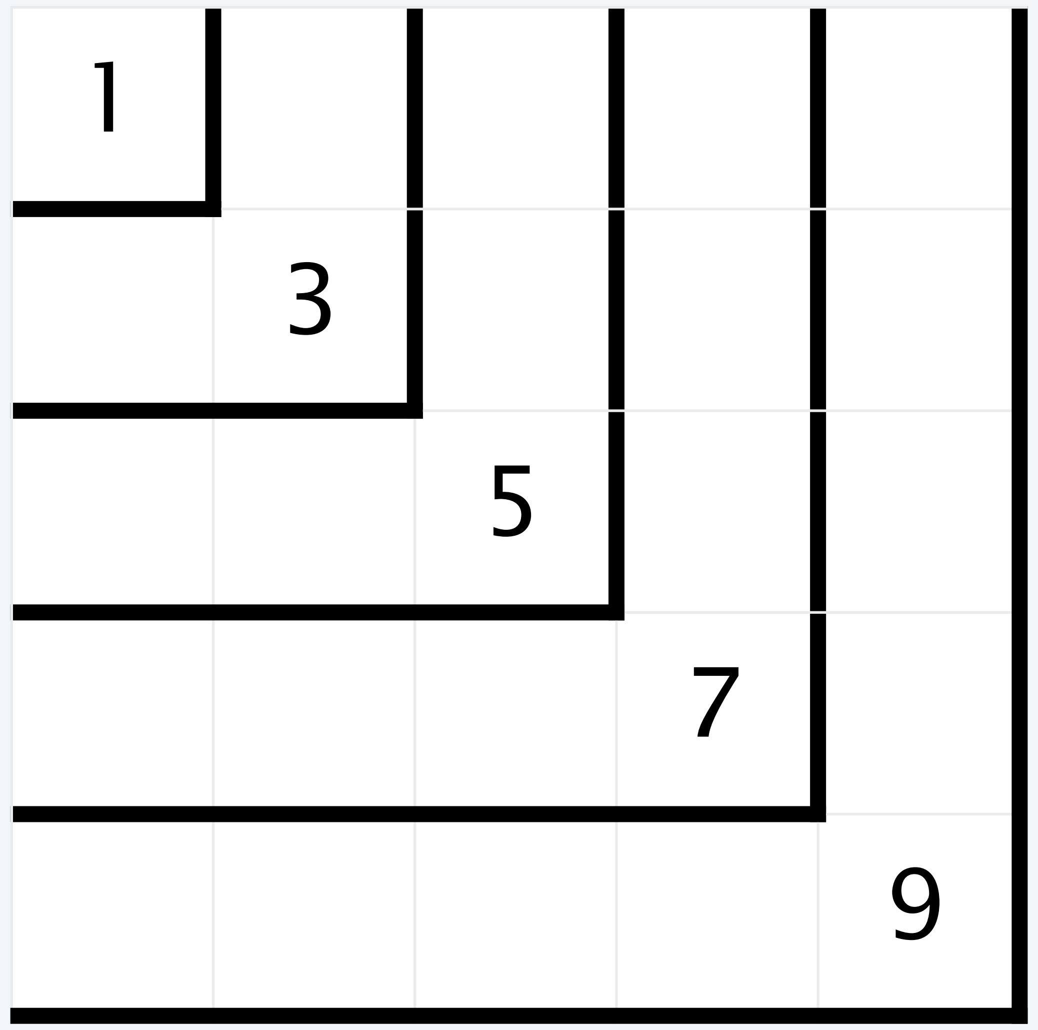
Example

The sum of the first N odd integers is N^2.

Base case. True for $N = 1$.

Induction step. The Nth odd integer is $2N - 1$. Let $T_N = 1 + 3 + 5 + ... + (2N - 1)$ be the sum of the first N odd integers.

- Assume that $T_{N-1} = (N - 1)^2$.
- Then $T_N = (N - 1)^2 + (2N - 1) = N^2$.



An alternate proof

Example: Convert an integer to binary

Recursive program

To compute a function of a positive integer *N*

- Base case. Return a value for small *N* .
- Reduction step. Assuming that it works for smaller values of its argument, use the function to compute a return value for *N*.

```
public class Binary
{
   public static String convert(int N)
   {
      if (N == 1) return "1";
      return convert(N/2) + (N % 2);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      StdOut.println(convert(N));
   }
}
```

int 0 or 1 automatically converted to String "0" or "1"

```
% java Binary 6
110
% java Binary 37
100101
% java Binary  999999
11110100001000111111
```

Q. How can we be convinced that this method is correct?

A. Use *mathematical induction*.

Proving a recursive program correct

Recursion

To compute a function of N

- Base case. Return a value for small N .
- Reduction step. Assuming that it works for smaller values of its argument, use the function to compute a return value for N.

Mathematical induction

To prove a statement involving N

- Base case. Prove it for small N.
- Induction step. Assuming that the statement is true for all positive integers less than N, use that fact to prove it for N.

Recursive program

```
public static String convert(int N)
{
   if (N == 1) return "1";
   return convert(N/2) + (N % 2);
}
```

Correctness proof, by induction

`convert()` computes the binary representation of N

- Base case. Returns "1" for $N = 1$.
- Induction step. Assume that `convert()` works for $N/2$

  1. Correct to append "0" if N is even, since $N = 2(N/2)$.

  2. Correct to append "1" if N is odd since $N = 2(N/2) + 1$.

Mechanics of a function call

System actions when *any* function is called

- *Save environment* (values of all variables and call location).
- *Initialize values* of argument variables.
- *Transfer control* to the function.
- *Restore environment* (and assign return value)
- *Transfer control* back to the calling code.

```
public class Binary
{
   public static String convert(int N)
   {
      if (N == 1) return "1";
      return convert(N/2) + (N % 2);
   }
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      System.out.println(convert(N));
   }
}
```

```
% java Convert 26
11010
```

```
convert(26)
   if (N == 1) return "1";
   return   "1101"    + "0";
```

```
convert(13)
   if (N == 1) return "1";
   return    "110"    + "1";
```

```
convert(6)
   if (N == 1) return "1";
   return     "11"    + "0";
```

```
convert(3)
   if (N == 1) return "1";
   return     "1"     + "1";
```

```
convert(1)
   if (N == 1) return "1";
   return convert(0) + "1";
```

Programming with recursion: typical bugs

Missing base case

```
public static double bad(int N)
{
   return bad(N-1) + 1.0/N;
}
```

No convergence guarantee

```
public static double bad(int N)
{
   if (N == 1) return 1.0;
   return bad(1 + N/2) + 1.0/N;
}
```

Try $N = 2$

Both lead to *infinite recursive loops* (bad news).

On the CLI, stop them with Control+C

Collatz Sequence

Collatz function of *N*.

- If *N* is 1, stop.
- If *N* is even, divide by 2.
- If *N* is odd, multiply by 3 and add 1.

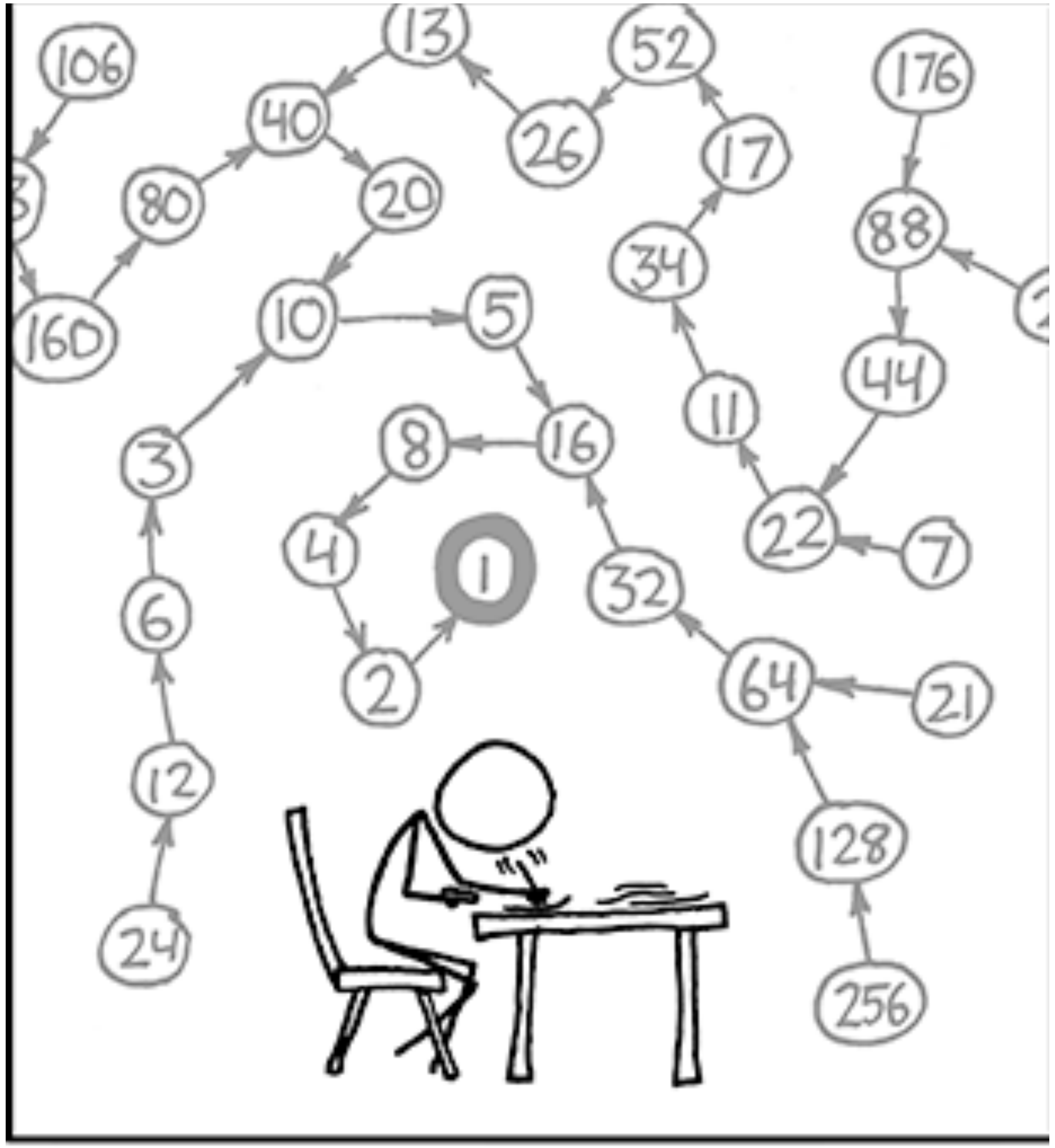
7	22	11	34	17	52	26	13	49	20	…

```
public static void collatz(int N)
{
    StdOut.print(N + " ");
    if (N == 1) return;
    if (N % 2 == 0) collatz(N / 2);
    else collatz(3*N + 1);
}
```

```
% java Collatz 7
7 22 11 34 17 52 26 13 40 20 10 5 16 8 4 2 1
```

Amazing fact. No one knows whether or not this function terminates for all *N* (!)

Note. We usually ensure termination by only making recursive calls for smaller *N*.



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

Image sources

http://xkcd.com/710/

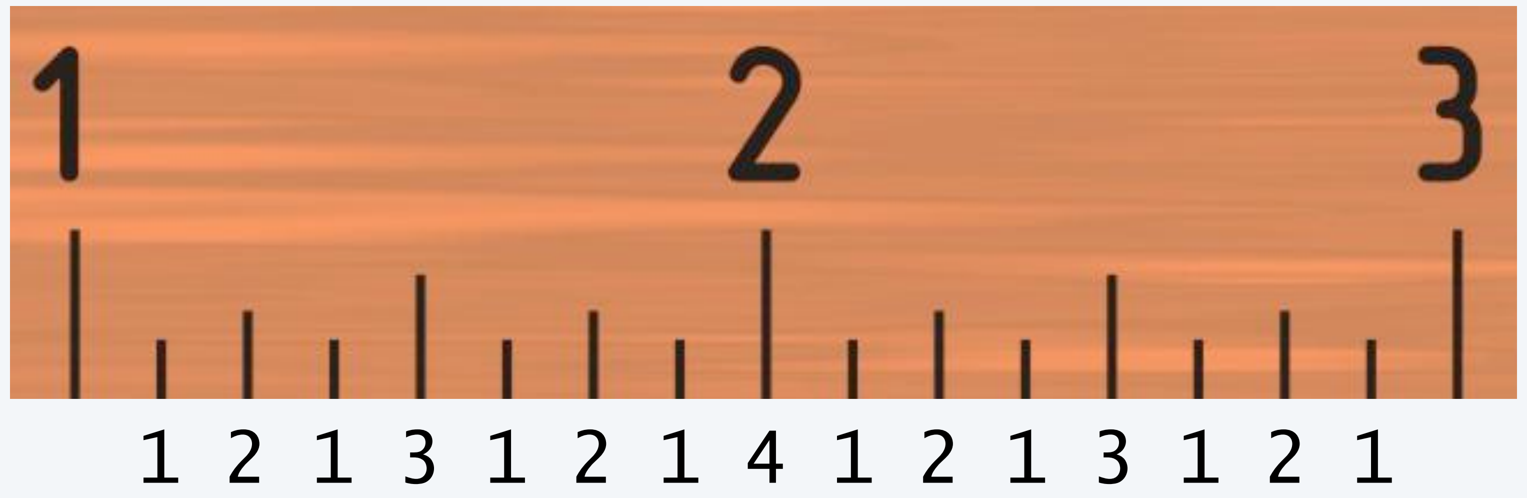
6. Recursion

- Foundations
- A classic example
- Recursive graphics
- Avoiding exponential waste
- Dynamic programming

Warmup: subdivisions of a ruler (revisited)

`ruler(n)`: create subdivisions of a ruler to $1/2^n$ inches.

- Return one space for $n = 0$.
- Otherwise, sandwich n between two copies of `ruler(n-1)`.




```
public class Ruler
{
   public static String ruler(int n)
   {
      if (n == 0) return " ";
      return ruler(n-1) + n + ruler(n-1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(ruler(n));
   }
}
```

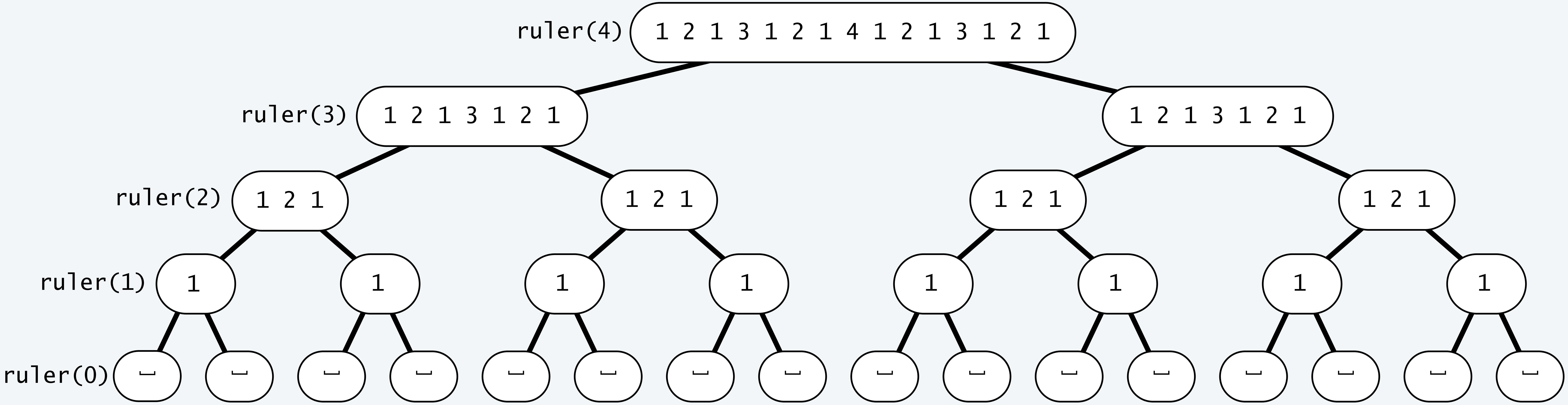
```
% java Ruler 1
 1
% java Ruler 2
 1 2 1
% java Ruler 3
 1 2 1 3 1 2 1
% java Ruler 4
 1 2 1 3 1 2 1 4 1 2 1 3 1 2 1
% java Ruler 50
Exception in thread "main"
java.lang.OutOfMemoryError:
Java heap space
```

$2^{50} - 1$ integers in output.

Tracing a recursive program

Use a *recursive call tree*

- One node for each recursive call.
- Label node with return value after children are labeled.



Towers of Hanoi puzzle

A legend of uncertain origin

- $n = 64$ discs of differing size; 3 posts; discs on one of the posts from largest to smallest.
- An ancient prophecy has commanded monks to move the discs to another post.
- When the task is completed, *the world will end.*

Rules

- Move discs one at a time.
- Never put a larger disc on a smaller disc.

Q. Generate list of instruction for monks ?

Q. When might the world end ?

n = 10

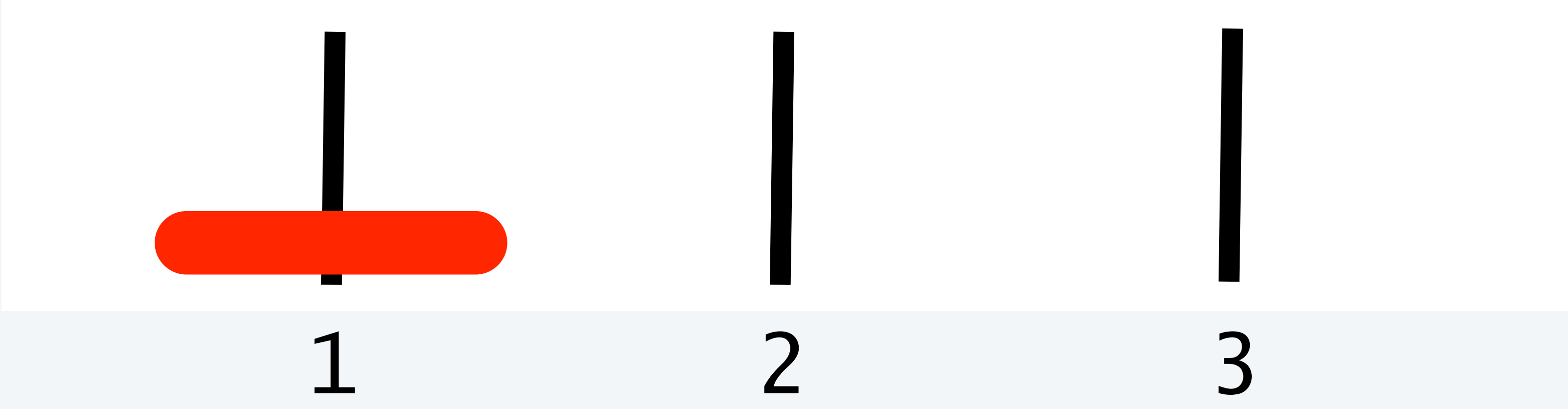
before

after

Towers of Hanoi

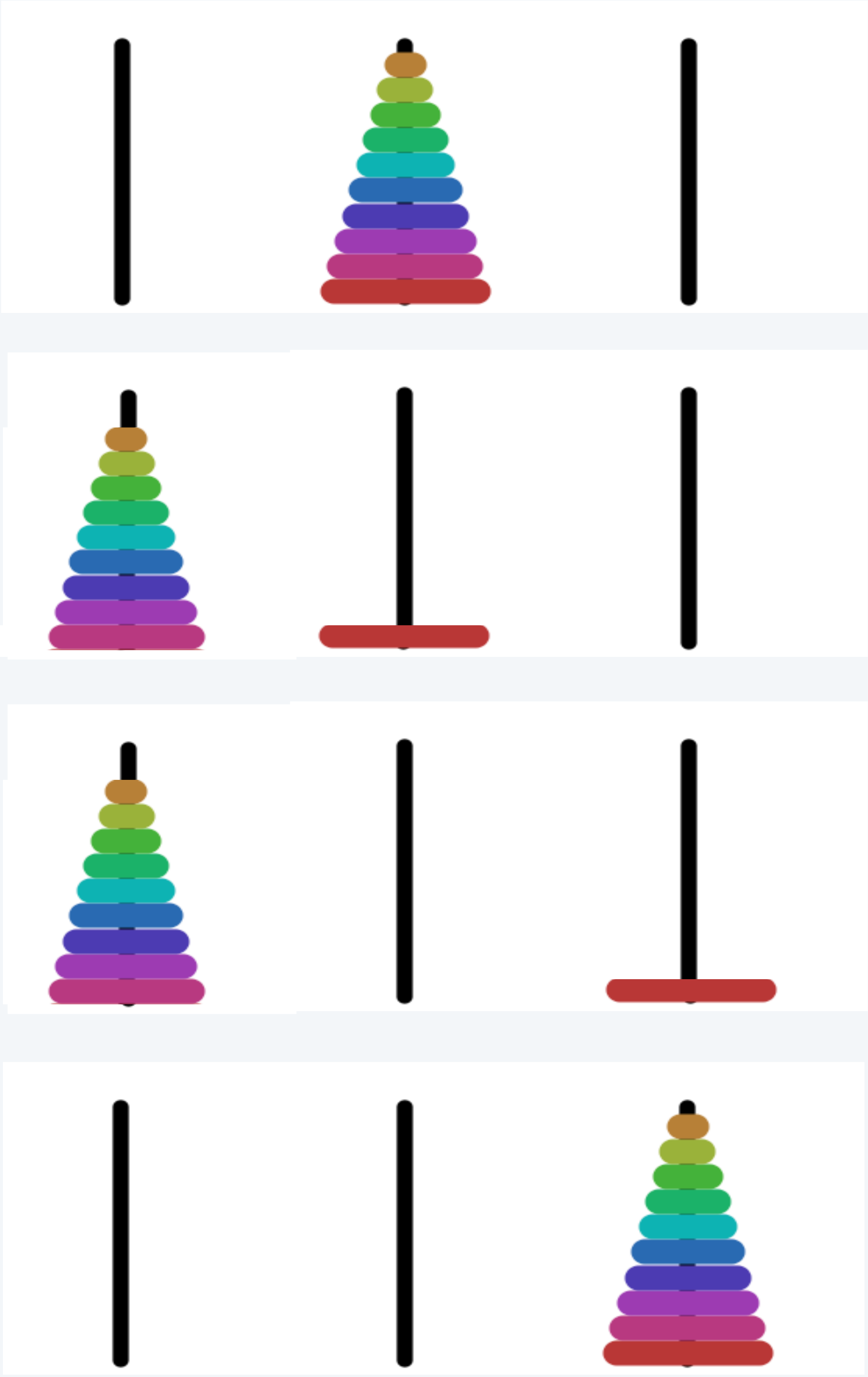
For simple instructions, use cyclic wraparound

- Move *right* means 1 to 2, 2 to 3, or 3 to 1.
- Move *left* means 1 to 3, 3 to 2, or 2 to 1.

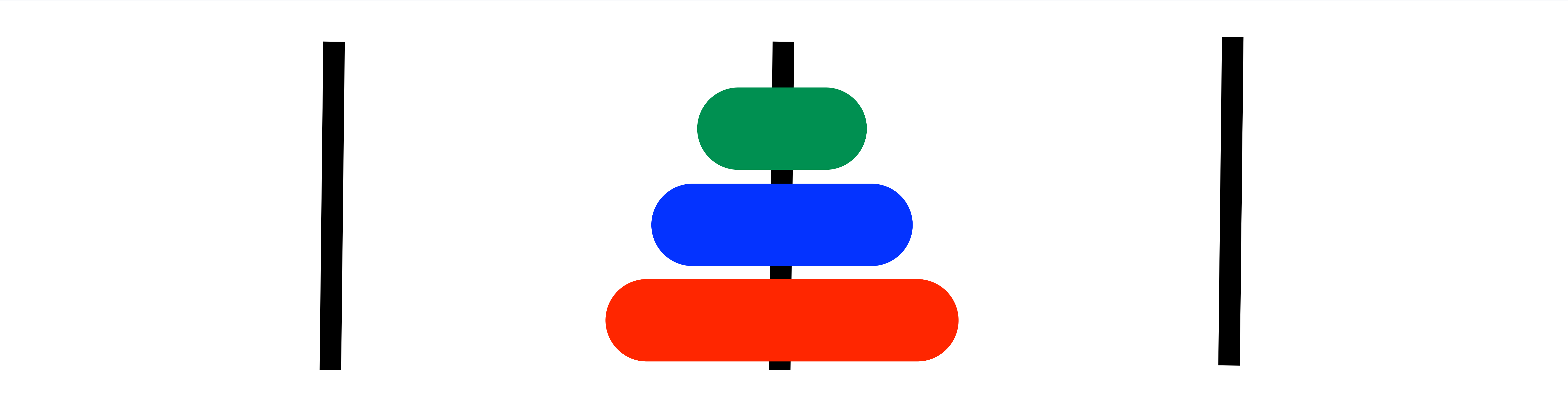


A recursive solution

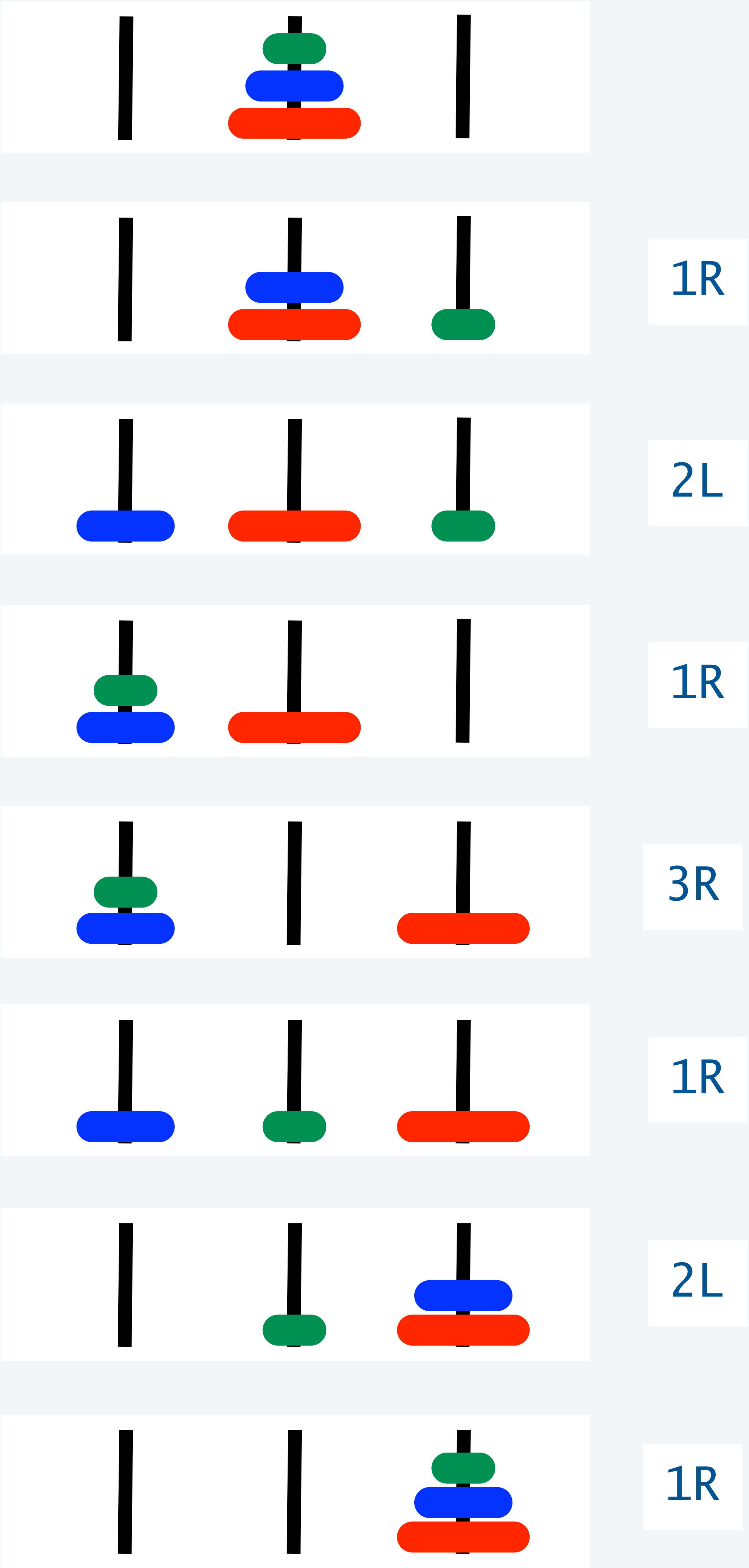
- Move $n - 1$ discs to the left (recursively).
- Move largest disc to the *right*.
- Move $n - 1$ discs to the left (recursively).



Towers of Hanoi solution (n = 3)



1R	2L	1R	3R	1R	2L	1R



Towers of Hanoi: recursive solution

`hanoi(n)`: Print moves for *n* discs.

- Return one space for $n = 0$.
- Otherwise, set `move` to the specified move for disc *n.*
- Then sandwich `move` between two copies of `hanoi(n-1)`.

```
public class Hanoi
{
   public static String hanoi(int n, boolean left)
   {
      if (n == 0) return " ";
      String move;
      if (left) move = n + "L";
      else      move = n + "R";
      return hanoi(n-1, !left) + move + hanoi(n-1, !left);
   }

   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(hanoi(n, false));
   }
}
```

```
% java Hanoi 3
 1R 2L 1R 3R 1R 2L 1R
```

Recursive call tree for towers of Hanoi

Structure is the *same* as for the ruler function and suggests 3 useful and easy-to-prove facts.

- Each disc always moves in the same direction.
- Moving smaller disc always alternates with a unique legal move.
- Moving n discs requires $2^n - 1$ moves.

Answers for towers of Hanoi

Q. Generate list of instructions for monks ?

A. (Long form). 1L 2R 1L 3L 1L 2R 1L 4R 1L 2R 1L 3L 1L 2R 1L 5L 1L 2R 1L 3L 1L 2R 1L 4R ...

A. (Short form). Alternate "1L" with the only legal move not involving the disc 1.

"L" or "R" depends on whether *n* is odd or even

Q. When might the world end ?

A. Not soon: need $2^{64} - 1$ moves.

Note: Recursive solution has been proven optimal.

moves per second	*end of world*
1	5.84 billion centuries
1 billion	5.84 centuries

CS.6.B.Recursion.Hanoi

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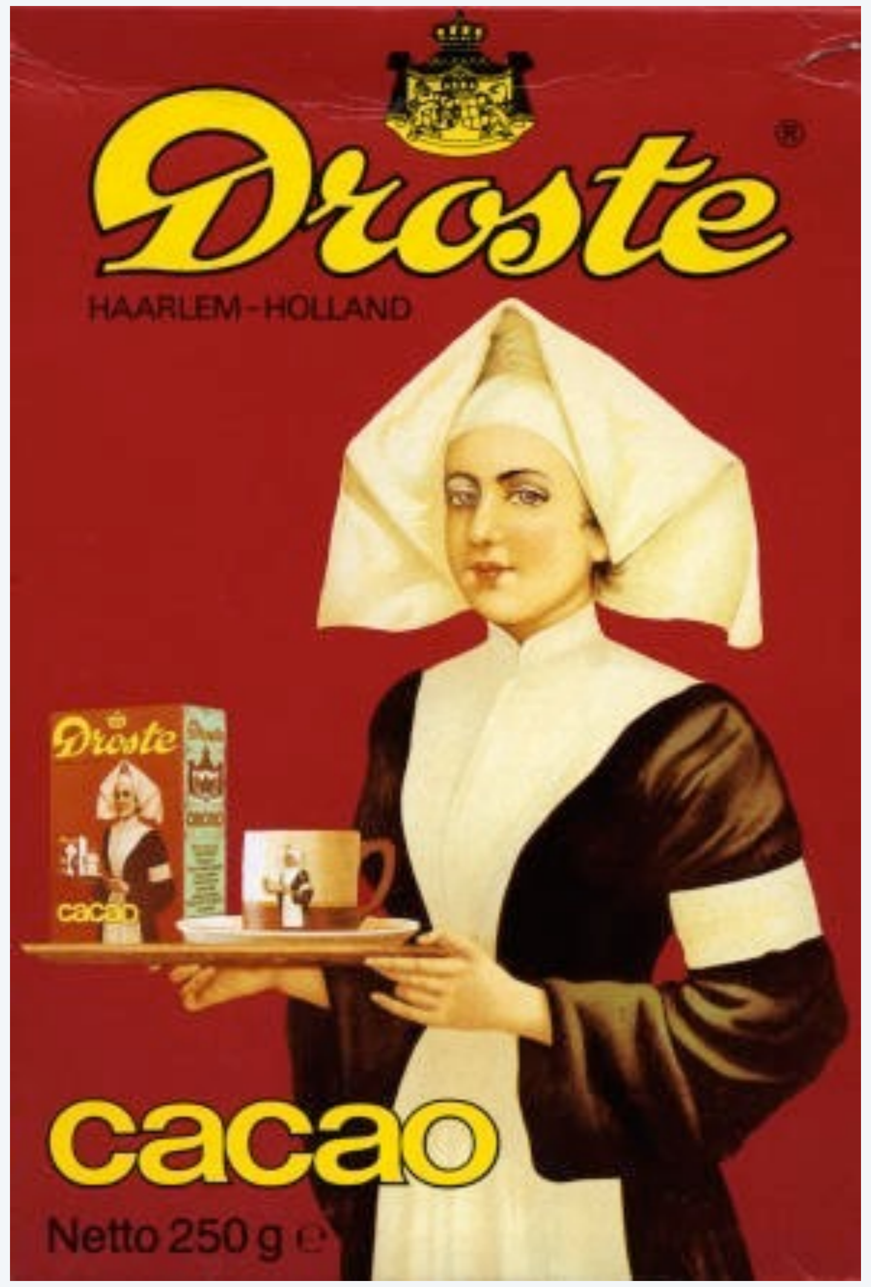
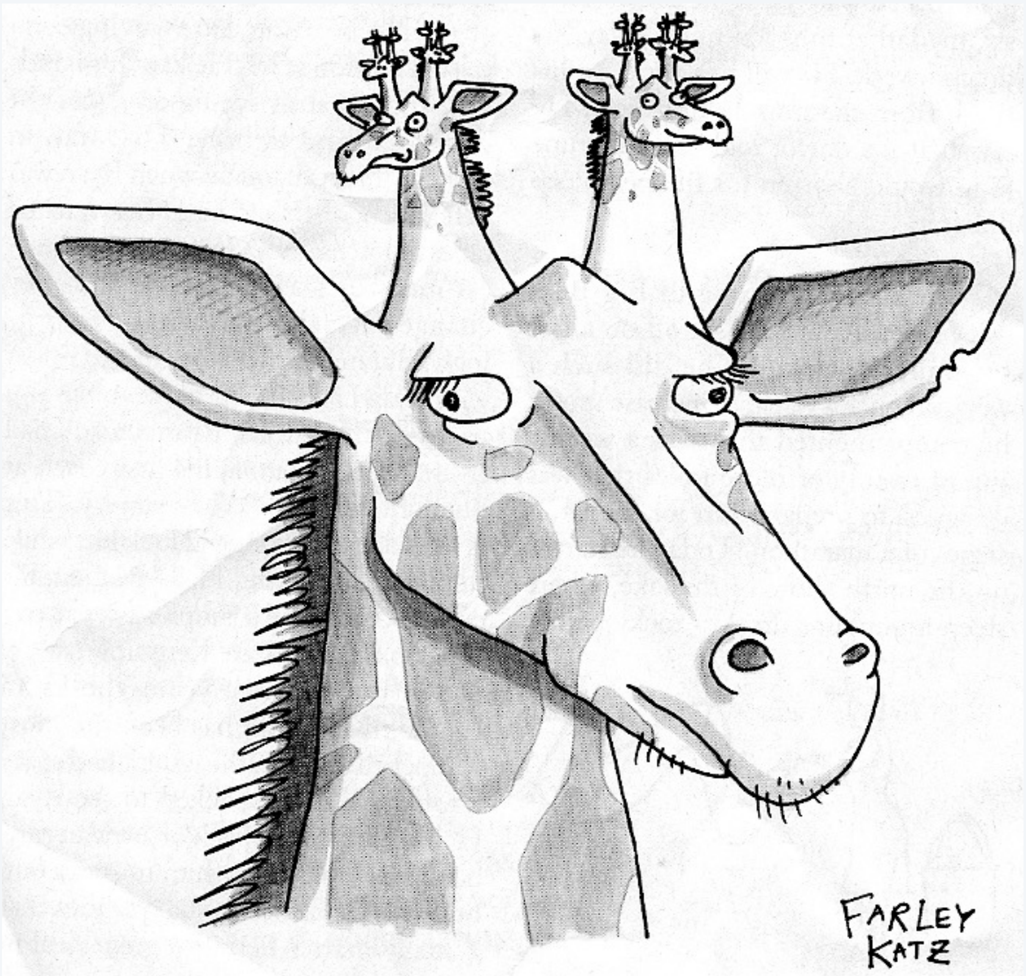
Recursive graphics in the wild






WEEKEND Arts FINE ARTS LEISURE

E41

FRIDAY, DECEMBER 15, 2006

The New York Times

Fruits of Design, Certified Organic

The Gifts to Open Again and Again

Black, White and Read All Over Over

By RANDY KENNEDY

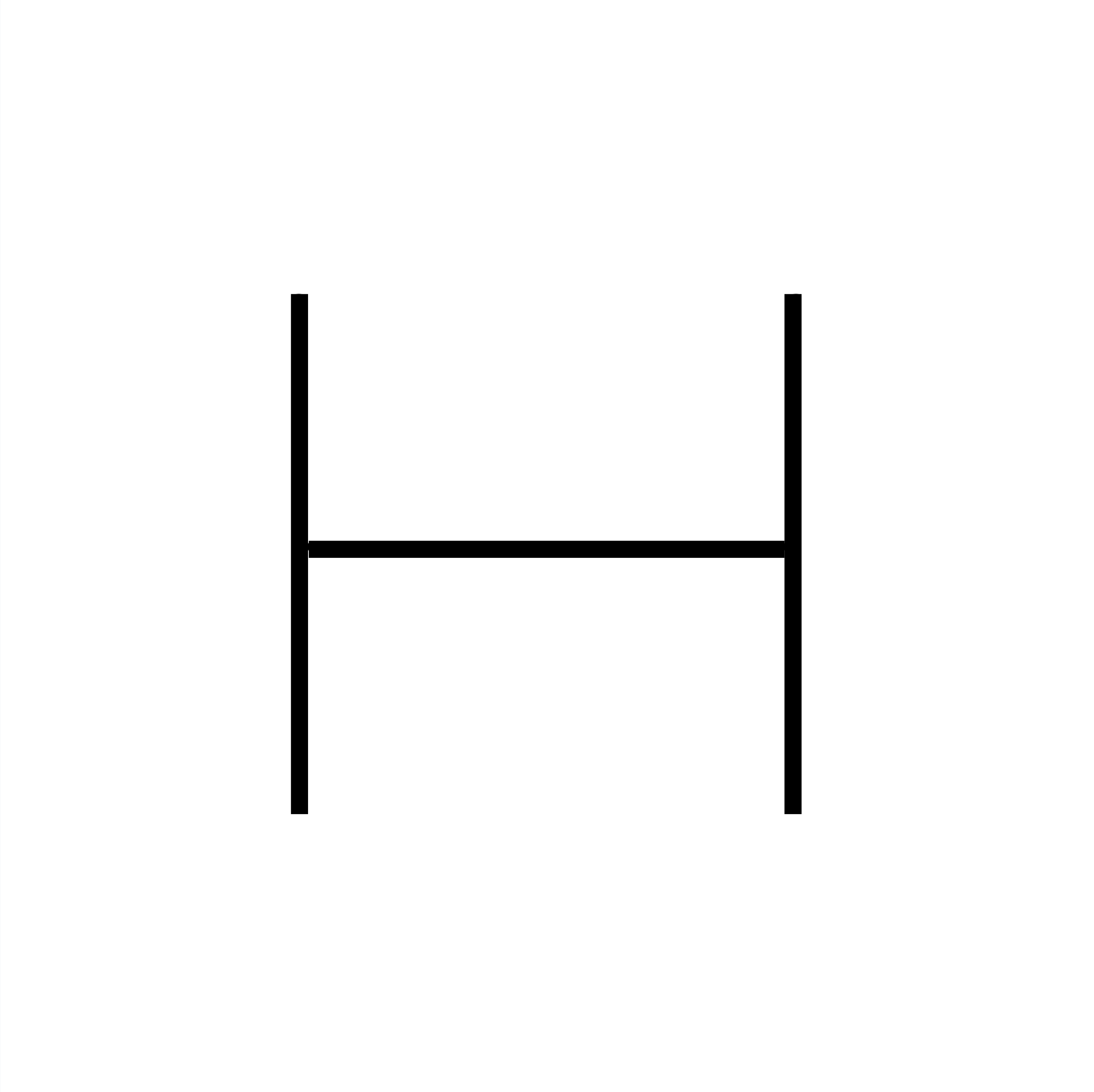
Divine and Devotee Meet Across Hinges

"Hello, World" of recursive graphics: H-trees

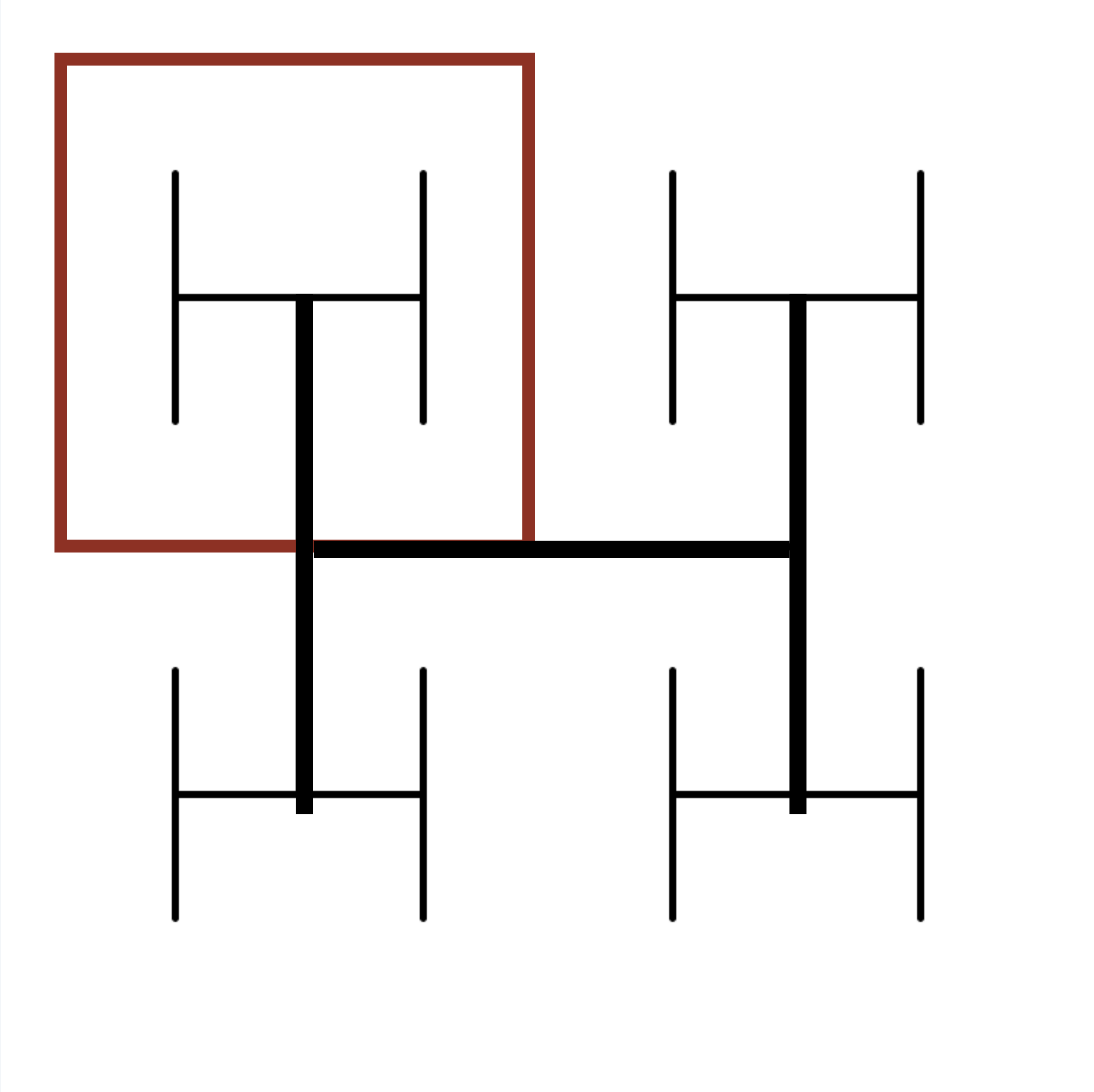
H-tree of order n

- If n is 0, do nothing.
- Draw an H, centered.
- Draw four H-trees of order $n-1$ and half the size, centered at the tips of the H.

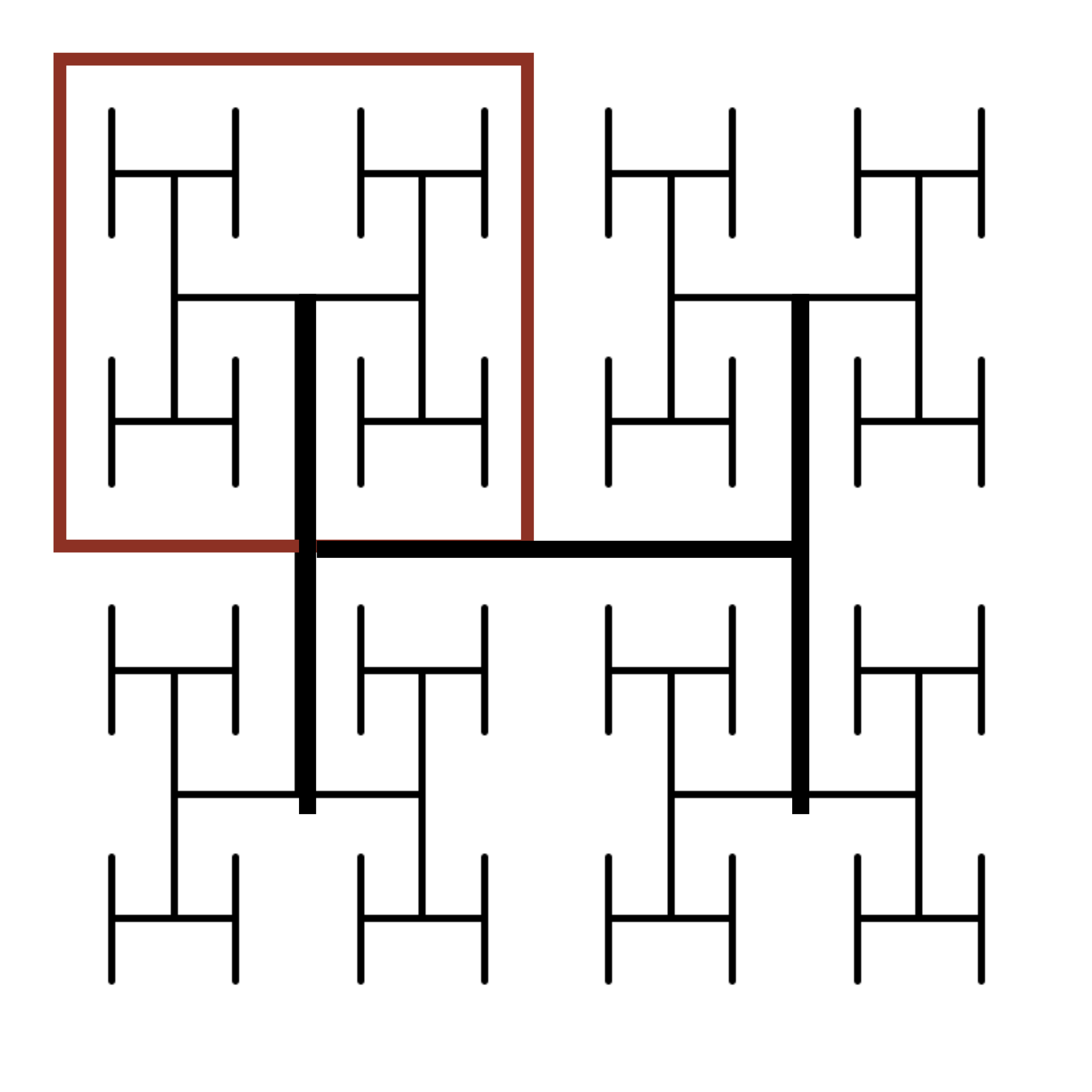
order 1



order 2

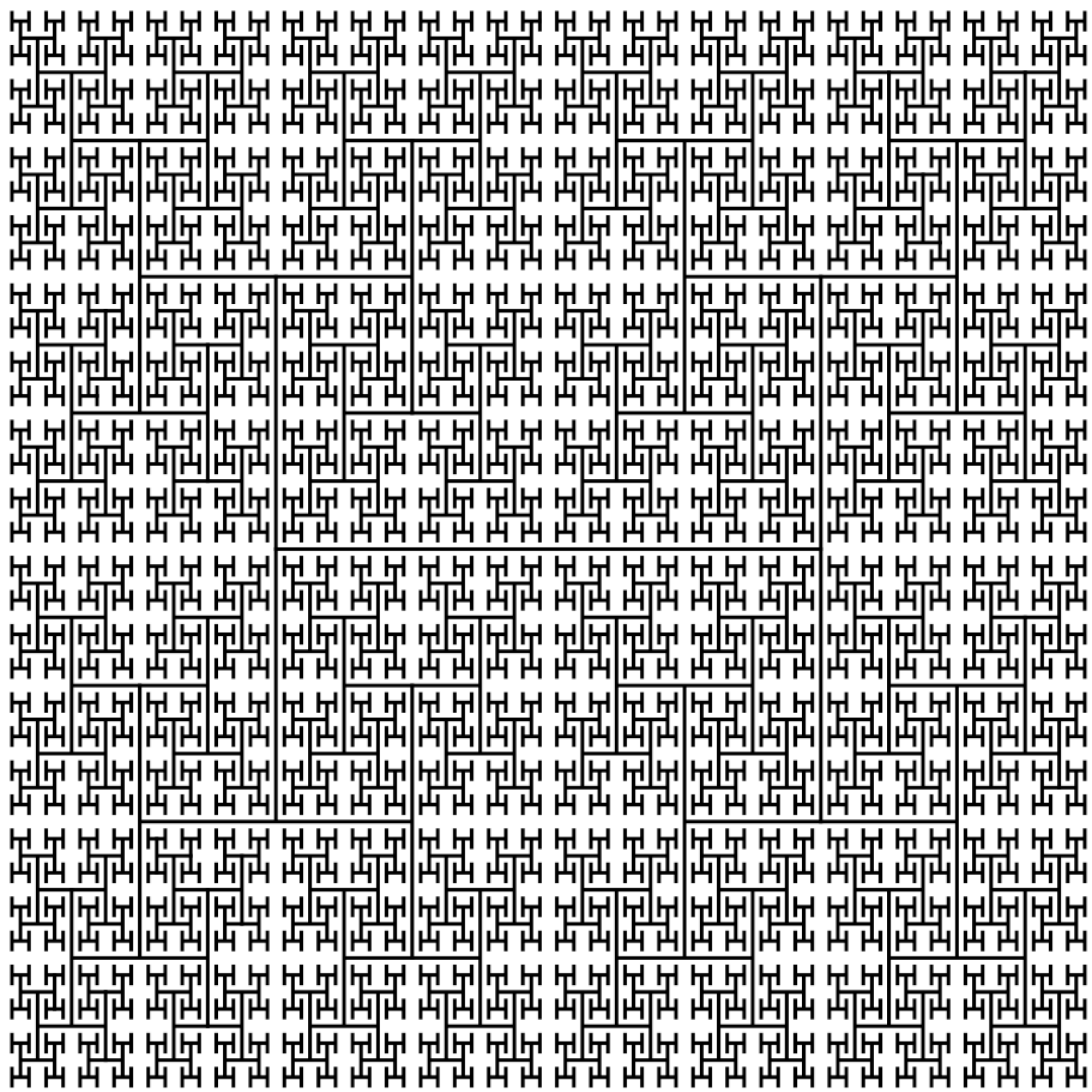


order 3



H-trees

Application. Connect a large set of regularly spaced sites to a single source.



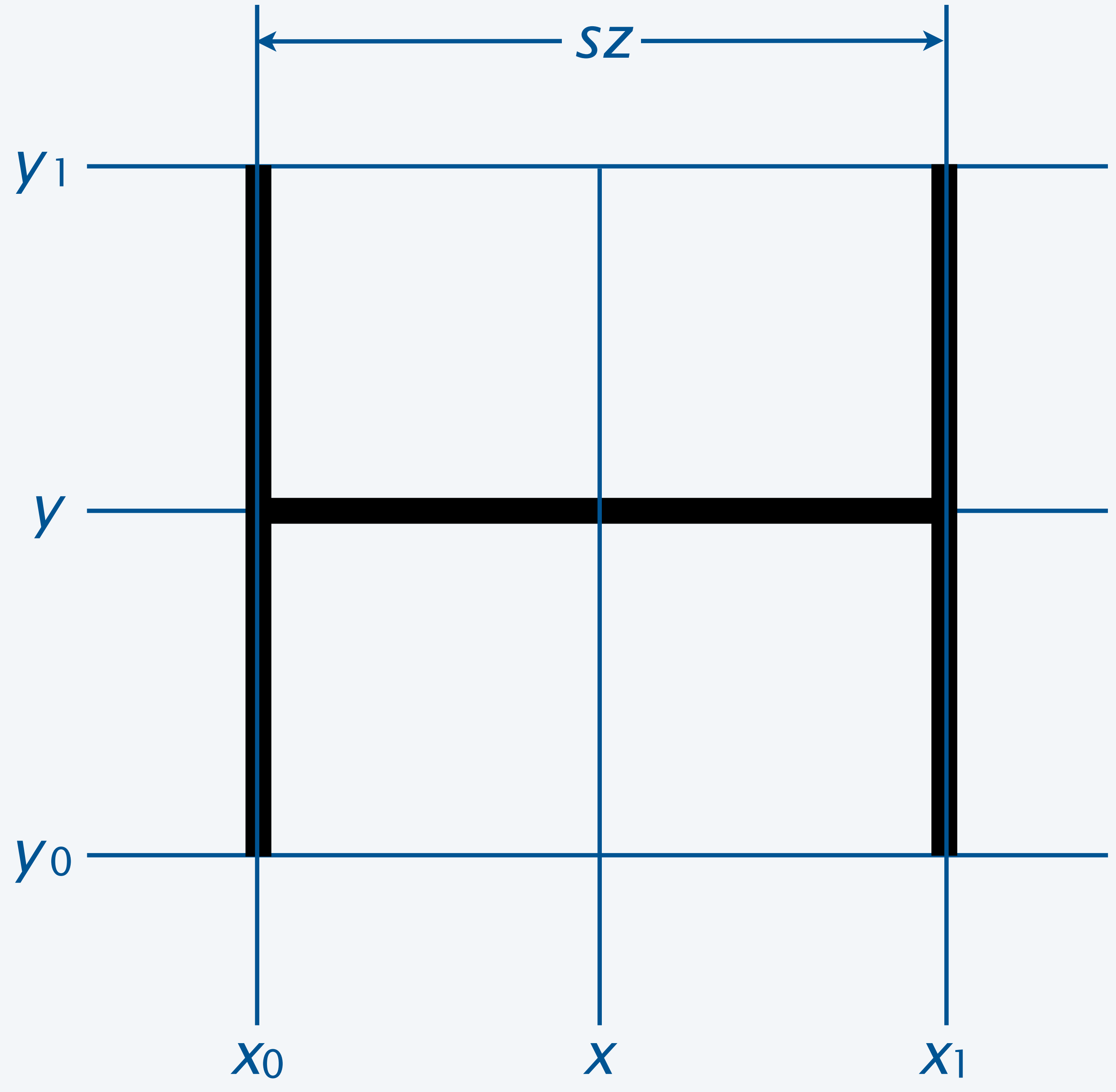
order 6

Recursive H-tree implementation

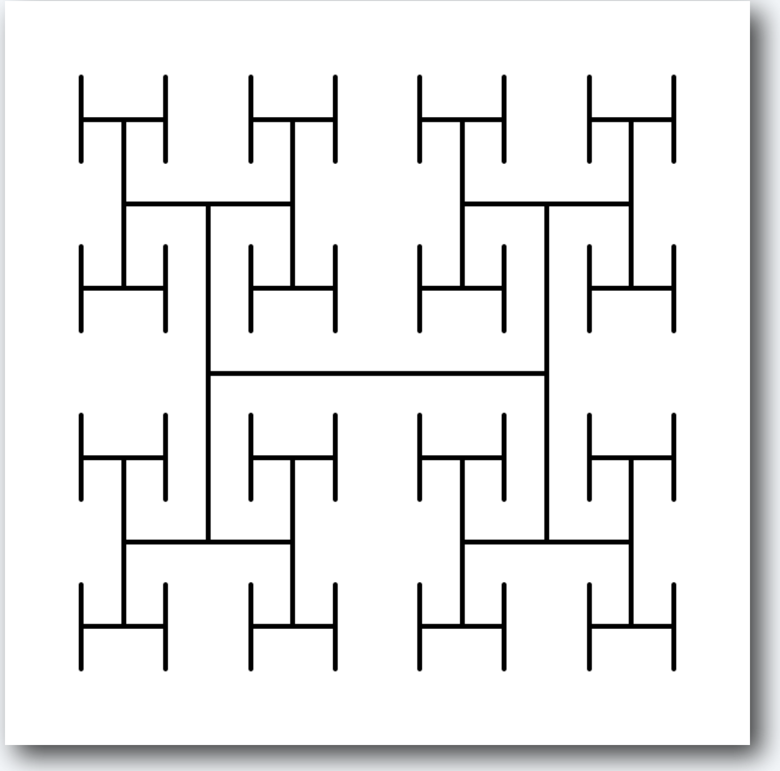
```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

draw the H, centered on (x, y)

draw four half-size H-trees



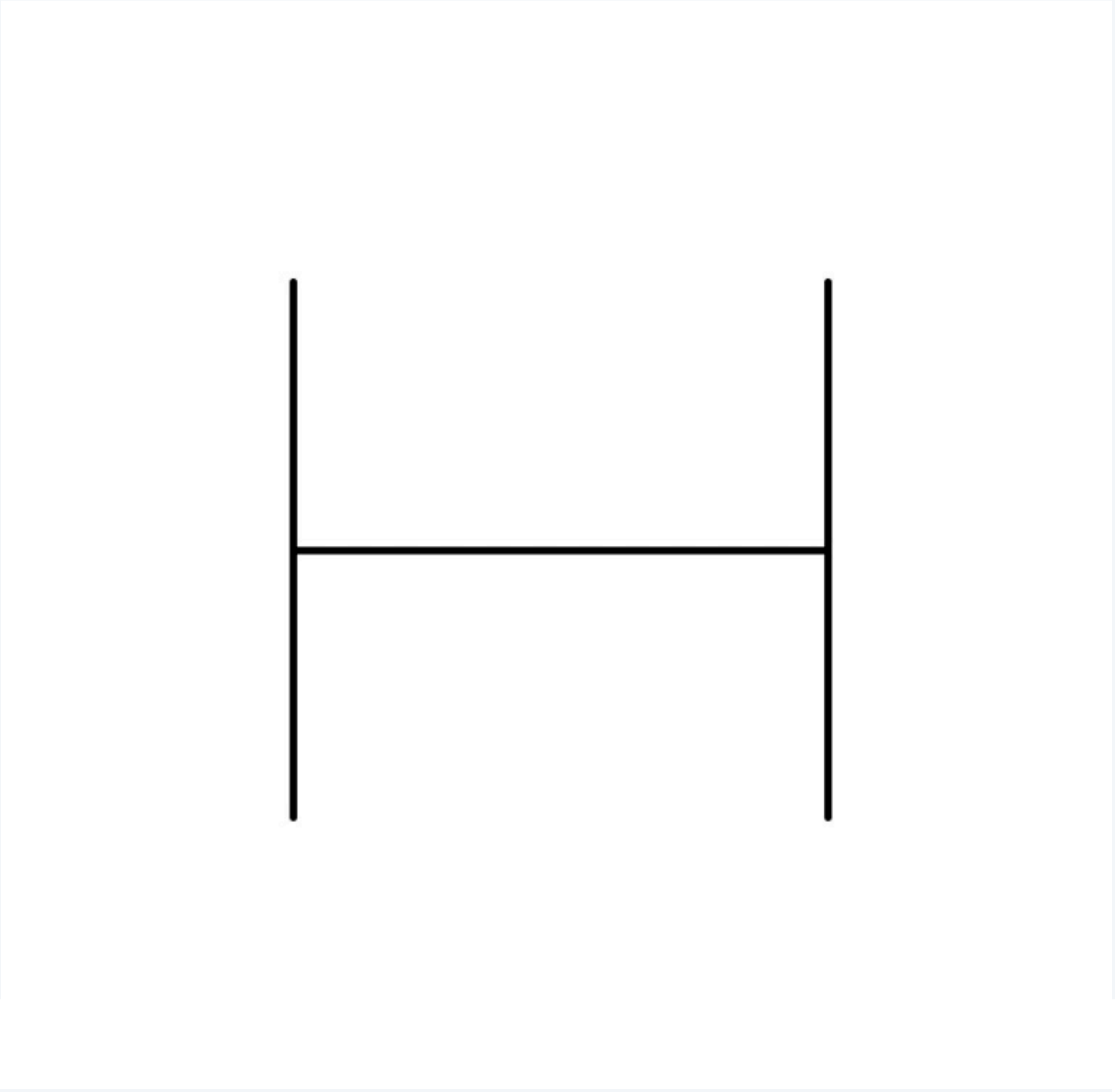

```
% java Htree 3
```



Deluxe H-tree implementation

```
public class HtreeDeluxe
{
   public static void draw(int n, double sz,
                                   double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;
      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);
      StdDraw.line(x1, y0, x1, y1);
      StdAudio.play(PlayThatNote.note(n, .25*n));
      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

```
% java HtreeDeluxe 4
```



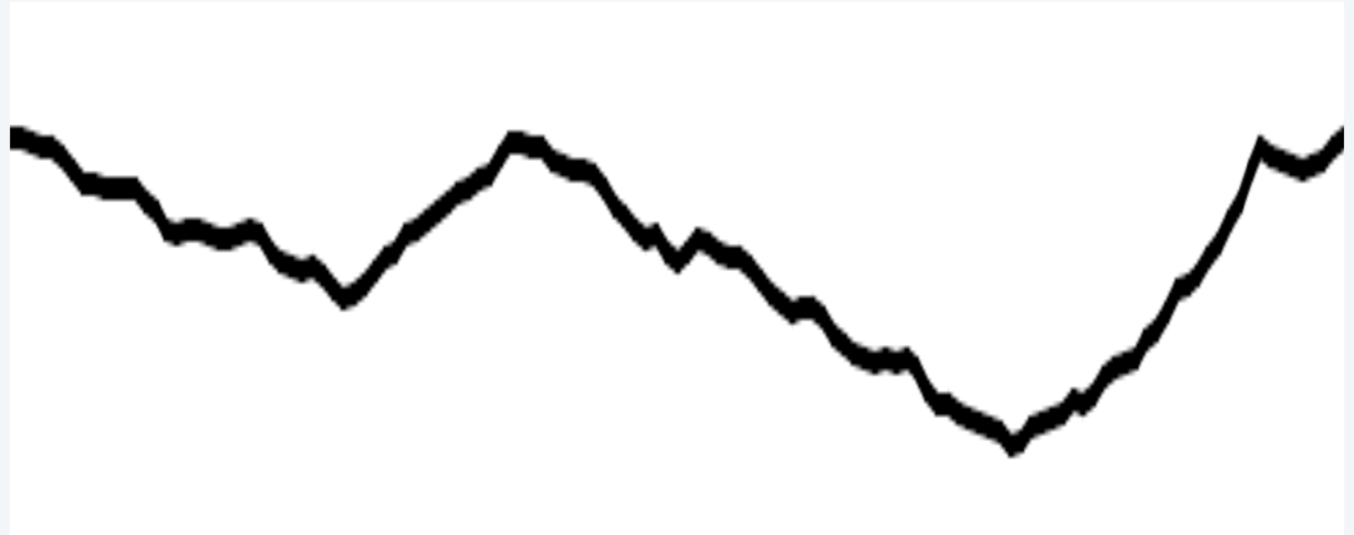
Fractional Brownian motion

A process that models many phenomenon.

- Price of stocks.
- Dispersion of fluids.
- Rugged shapes of mountains and clouds.
- Shape of nerve membranes.

. . .

Brownian bridge model



An actual mountain

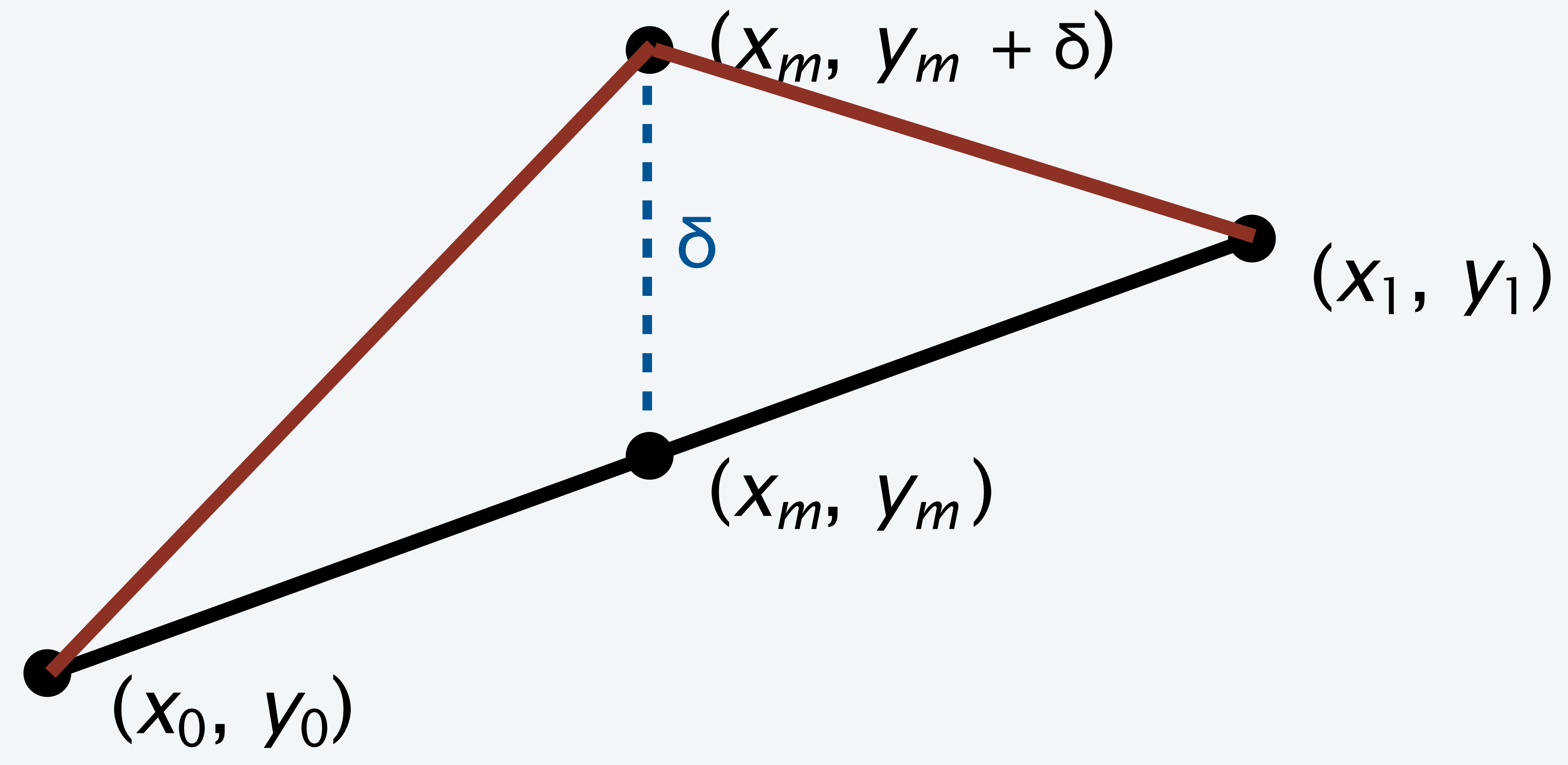
Price of an actual stock

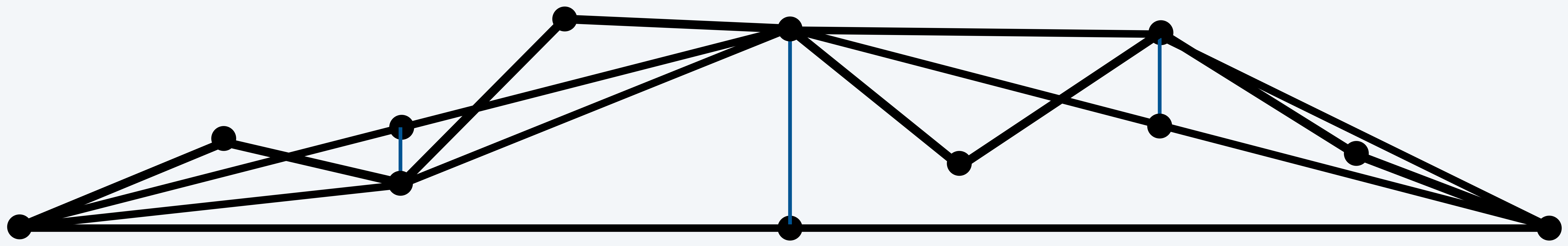
Black-Scholes model (two different parameters)

Fractional Brownian motion simulation

Midpoint displacement method

- Consider a line segment from (x_0, y_0) to (x_1, y_1).
- If sufficiently short draw it *and return.* Otherwise:
- Divide the line segment in half, at (x_m, y_m).
- Choose δ at random *from Gaussian distribution.*
- Add δ to y_m.
- Recur on the left and right line segments.



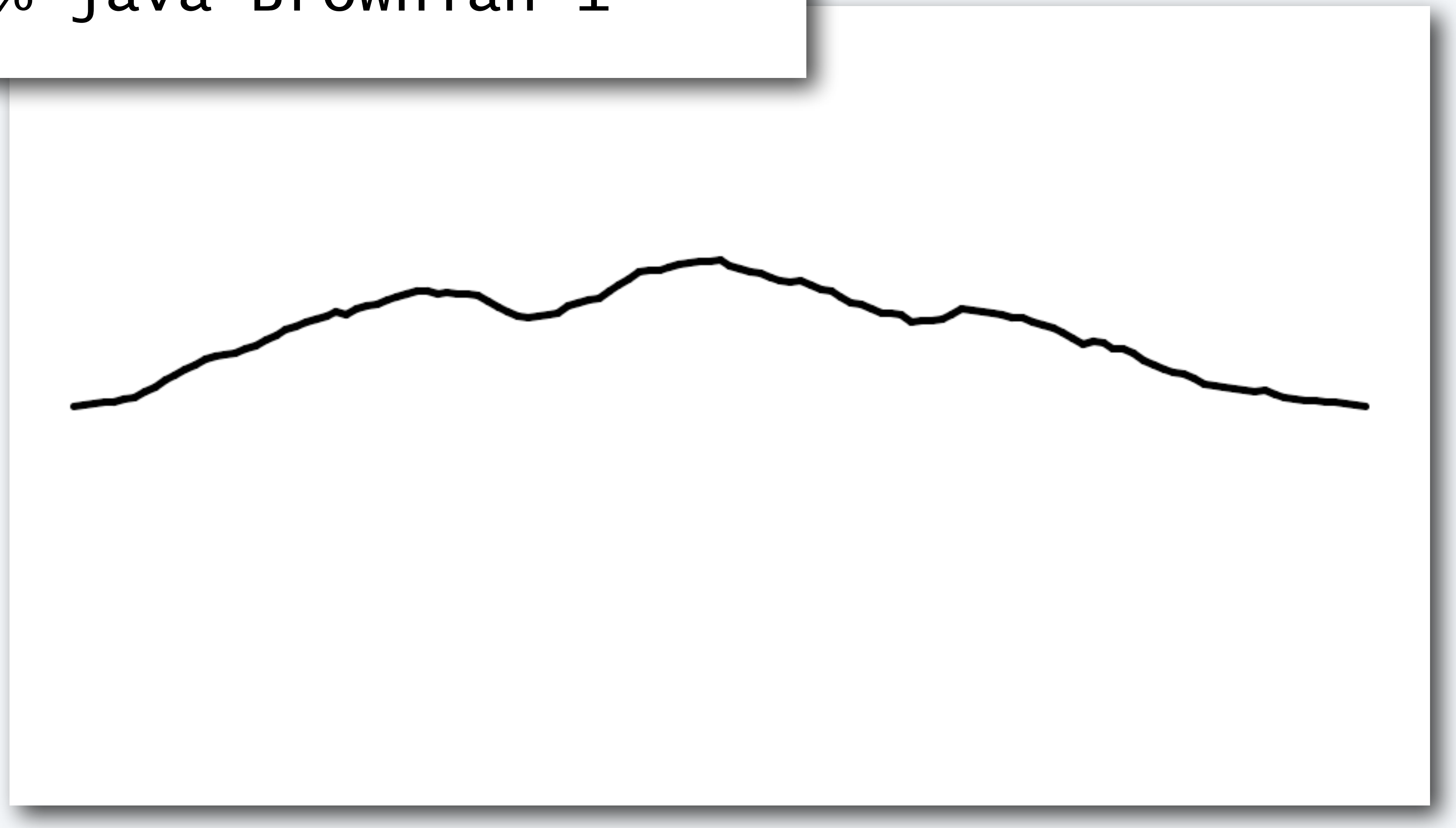


Brownian motion implementation

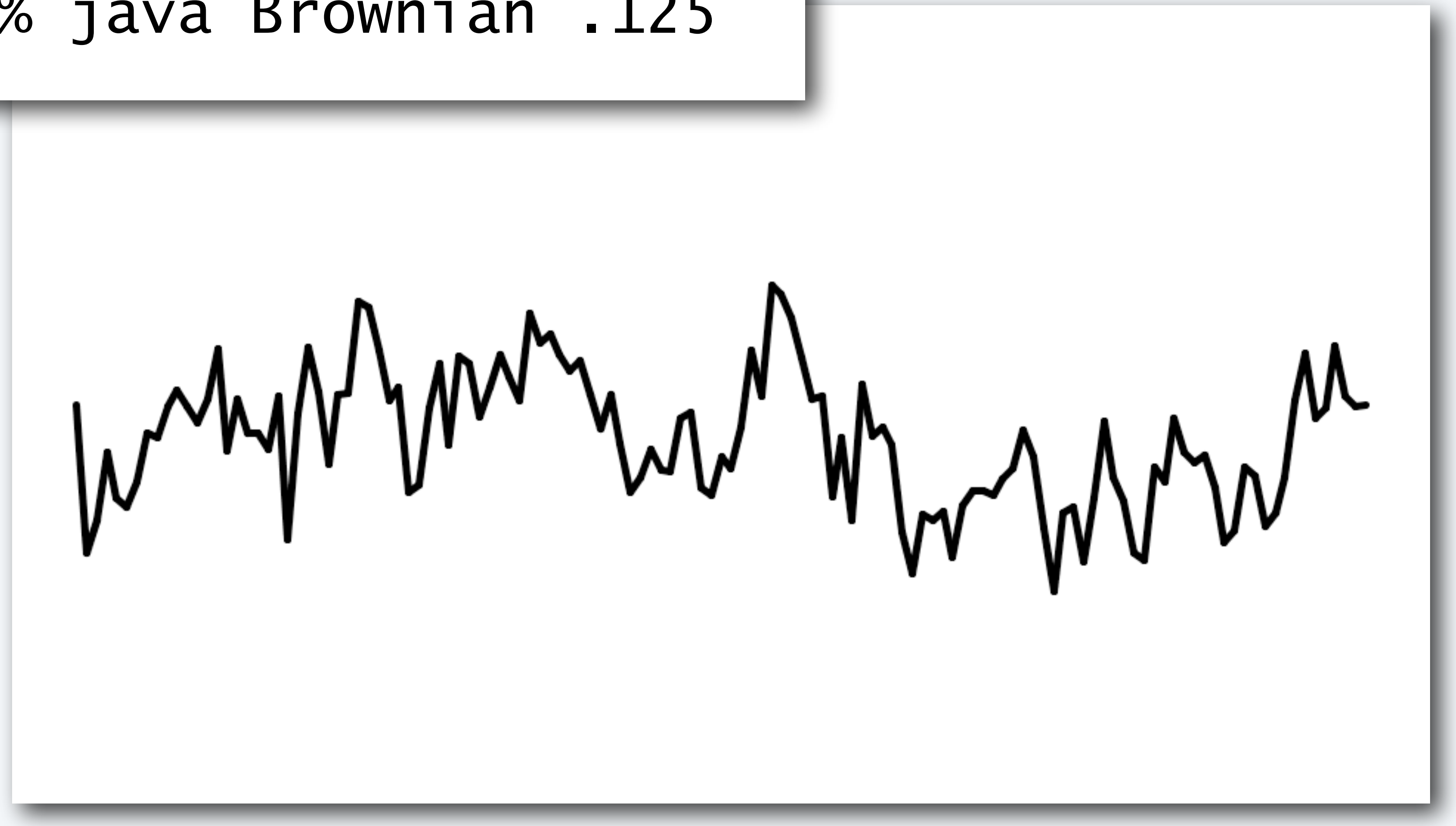
```
public class Brownian
{
   public static void
   curve(double x0, double y0, double x1, double y1,
                              double var, double s)
   {
      if (x1 - x0 < .01)
      {  StdDraw.line(x0, y0, x1, y1); return;  }
      double xm = (x0 + x1) / 2;
      double ym = (y0 + y1) / 2;
      double stddev = Math.sqrt(var);
      double delta = StdRandom.gaussian(0, stddev);
      curve(x0, y0, xm, ym+delta, var/s, s);
      curve(xm, ym+delta, x1, y1, var/s, s);
   }

   public static void main(String[] args)
   {
      double hurst = Double.parseDouble(args[0]);
      double s = Math.pow(2, 2*hurst);
      curve(0, .5, 1.0, .5, .01, s);
   }
}
```

control parameter (see text)

```
% java Brownian 1
```



```
% java Brownian .125
```



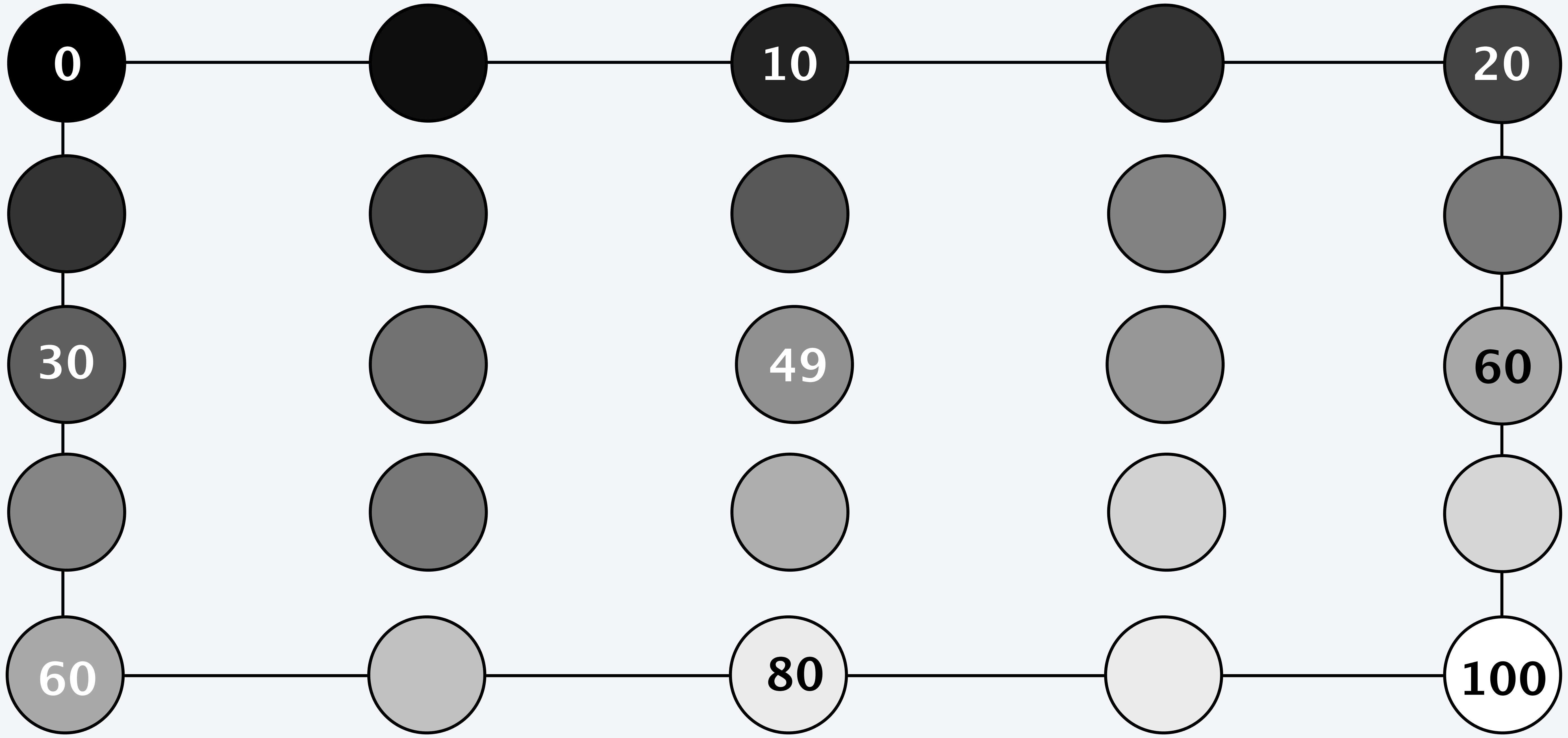
A 2D Brownian model: plasma clouds

Midpoint displacement method

- Consider a rectangle centered at (x, y) with pixels at the four corners.
- If the rectangle is small, do nothing. Otherwise:
- Color the midpoints of each side the average of the endpoint colors.
- Choose δ at random *from Gaussian distribution*.
- Color the center pixel the average of the four corner colors *plus* δ

Booksite code actually draws a rectangle to avoid artifacts

- Recurse on the four quadrants.



A Brownian cloud

A Brownian landscape

Image sources

http://en.wikipedia.org/wiki/Droste_effect#mediaviewer/File:Droste.jpg

http://www.mcescher.com/gallery/most-popular/circle-limit-iv/

http://www.megamonalisa.com/recursion/

http://fractalfoundation.org/OFC/FractalGiraffe.png

http://www.nytimes.com/2006/12/15/arts/design/15serk.html?pagewanted=all&_r=0

http://www.geocities.com/aaron_torpy/gallery.htm

START RECORDING

Attendance Quiz

Attendance Quiz: Recursion

- Scan the QR code, or find today's attendance quiz under the "Quizzes" tab on Canvas
- Password: to be announced in class
- After five minutes, we will discuss the answers

Note that the Fibonacci sequence is defined as:

Let $F_n = F_{n-1} + F_{n-2}$ for $n > 1$ with $F_0 = 0$ and $F_1 = 1$.

For example:

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13

Attendance Quiz: Recursion

- Write your name
- Complete the following Java program, FibonacciR.java
- Briefly explain why this naïve implementation will be slow, and how it can be improved.

Note that the Fibonacci sequence is defined as:

Let $F_n = F_{n-1} + F_{n-2}$ for $n > 1$ with $F_0 = 0$ and $F_1 = 1$.

For example:

n	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13

```
public class FibonacciR
{
   public static long F(int n)
   {
     # YOUR CODE GOES HERE
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(F(n));
   }
}
```

6. Recursion

- Foundations
- A classic example
- Recursive graphics
- **Avoiding exponential waste**
- Dynamic programming

Fibonacci numbers

Let $F_n = F_{n-1} + F_{n-2}$ for $n > 1$ with $F_0 = 0$ and $F_1 = 1$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	…
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	…

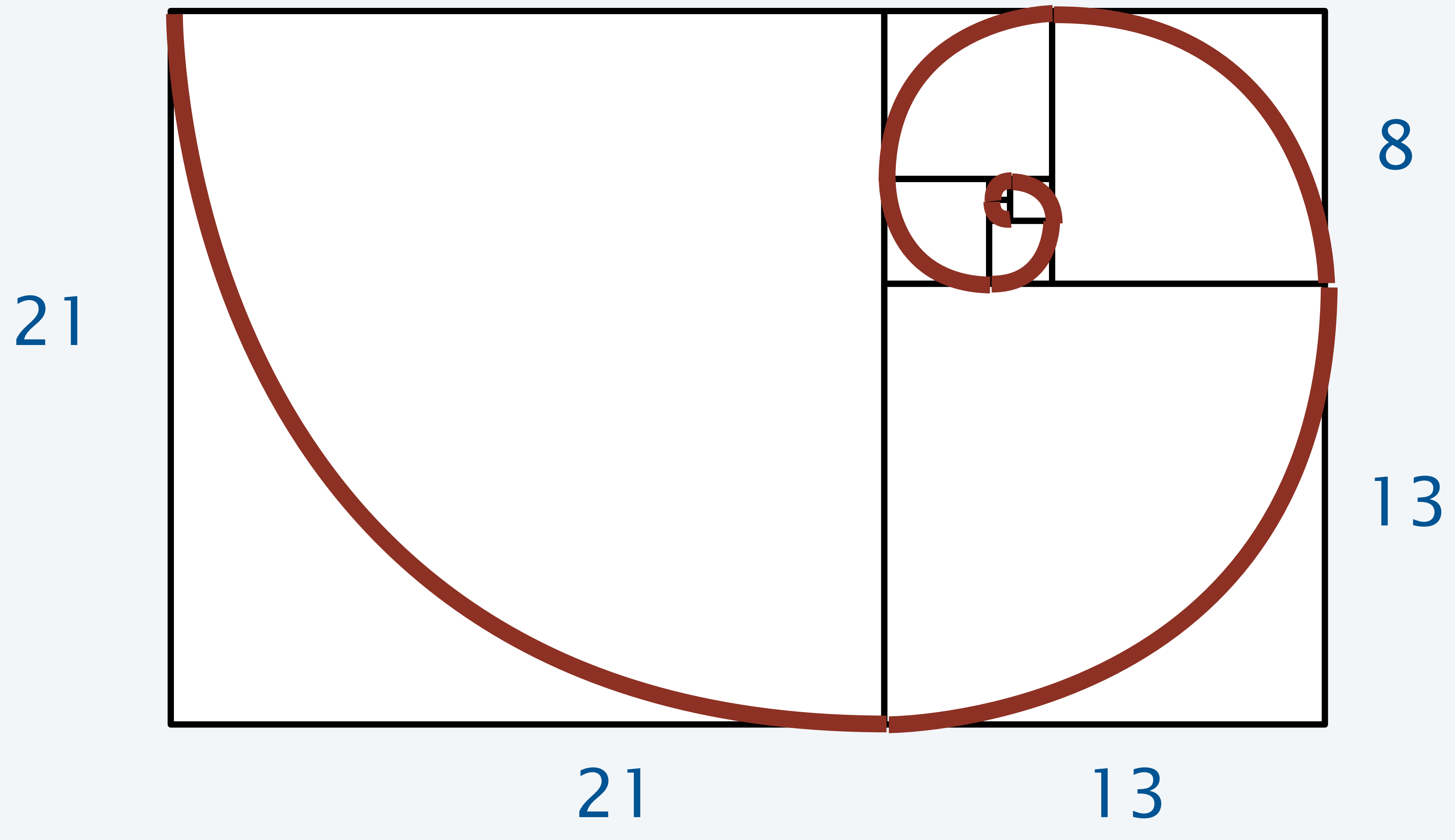
Leonardo Fibonacci
c. 1170 – c. 1250

Models many natural phenomena and is widely found in art and architecture.

Examples.

- Model for reproducing rabbits.
- Nautilus shell.
- Mona Lisa.
- …

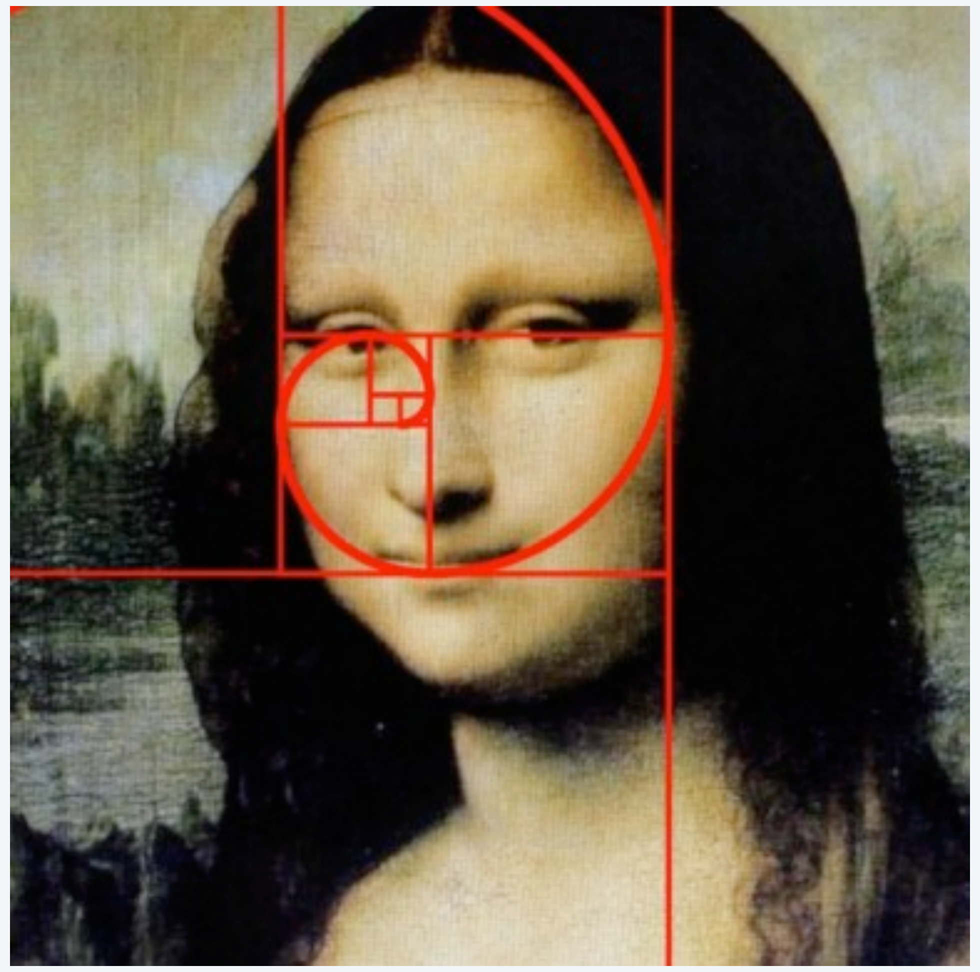
golden ratio F_n / F_{n-1}

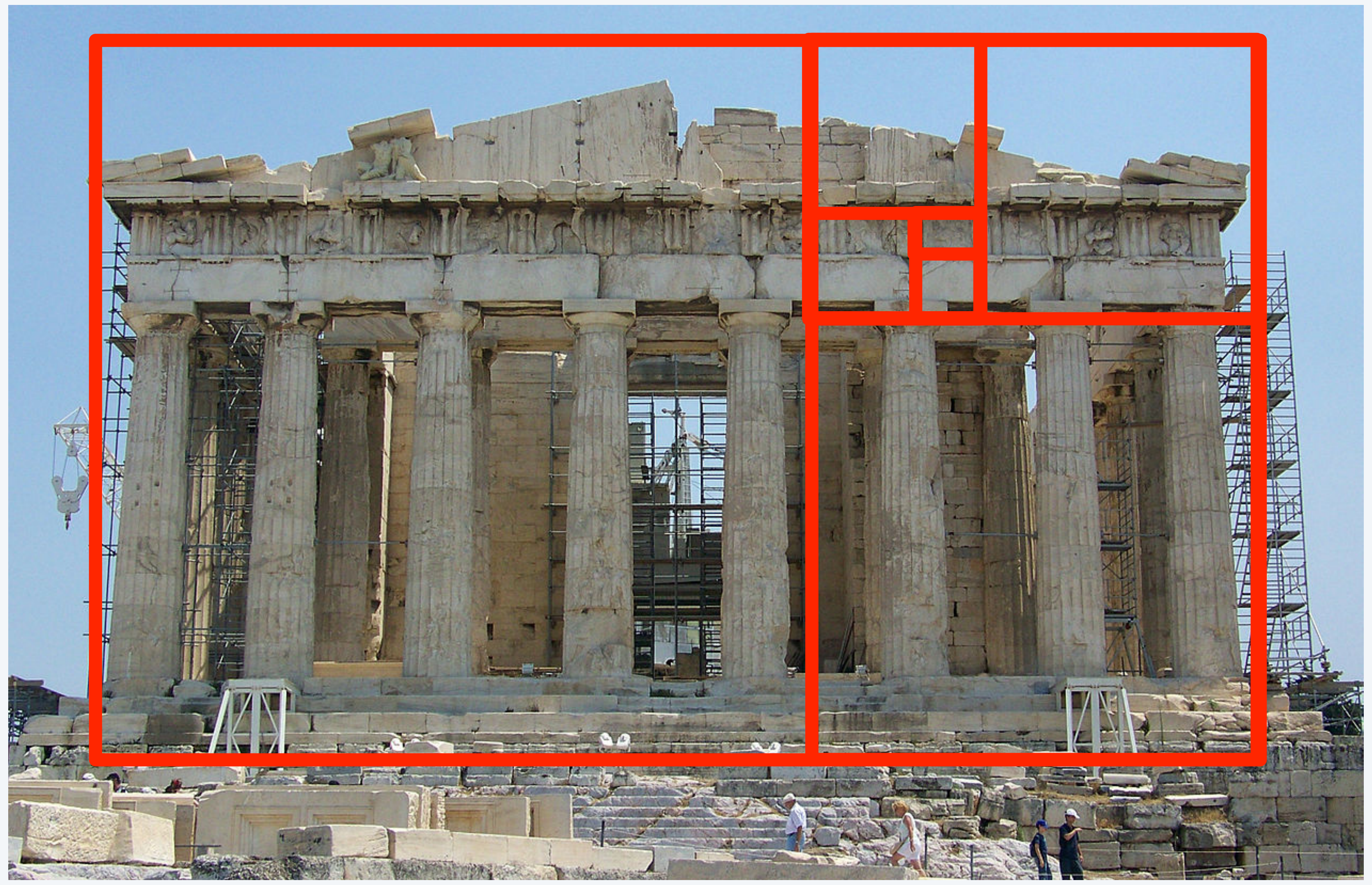


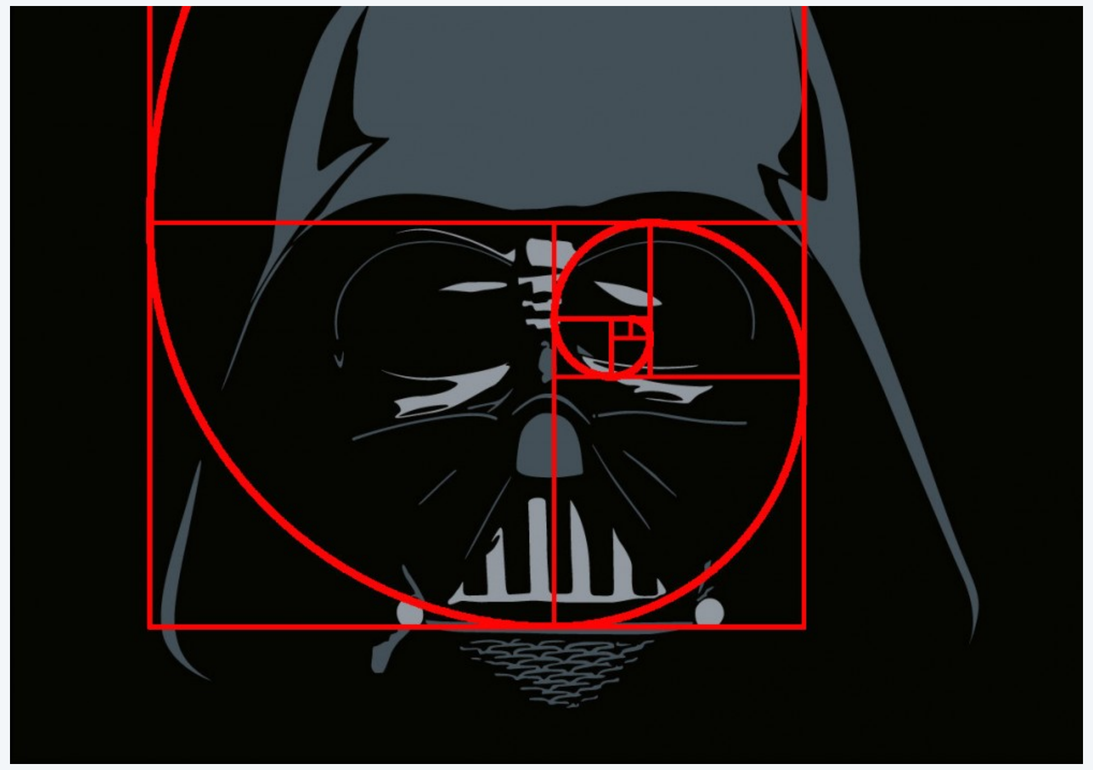
Facts (known for centuries).

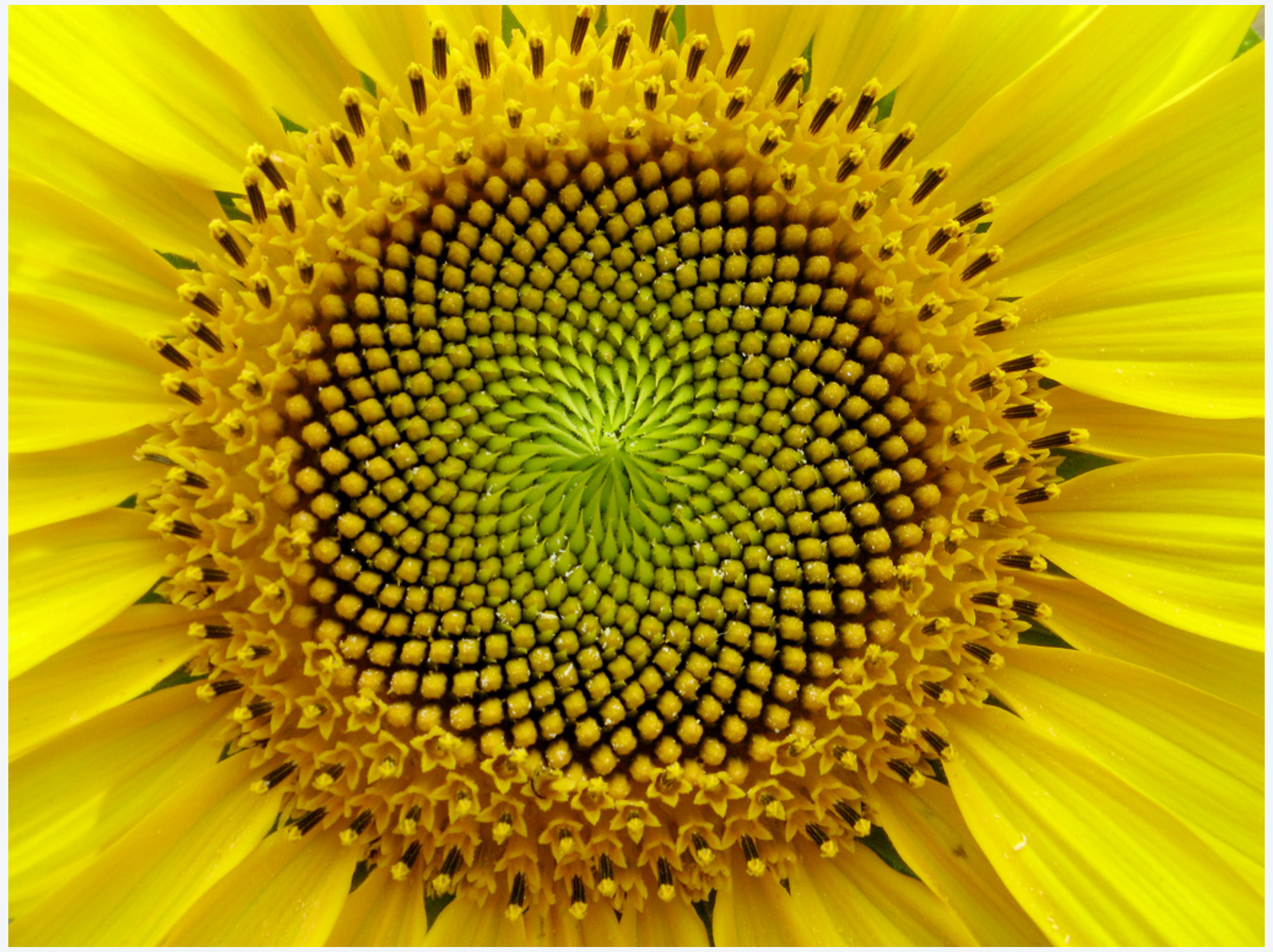
- $F_n / F_{n-1} \rightarrow \Phi = 1.618\ldots$ as $n \rightarrow \infty$
- F_n is the closest integer to $\Phi^n/\sqrt{5}$

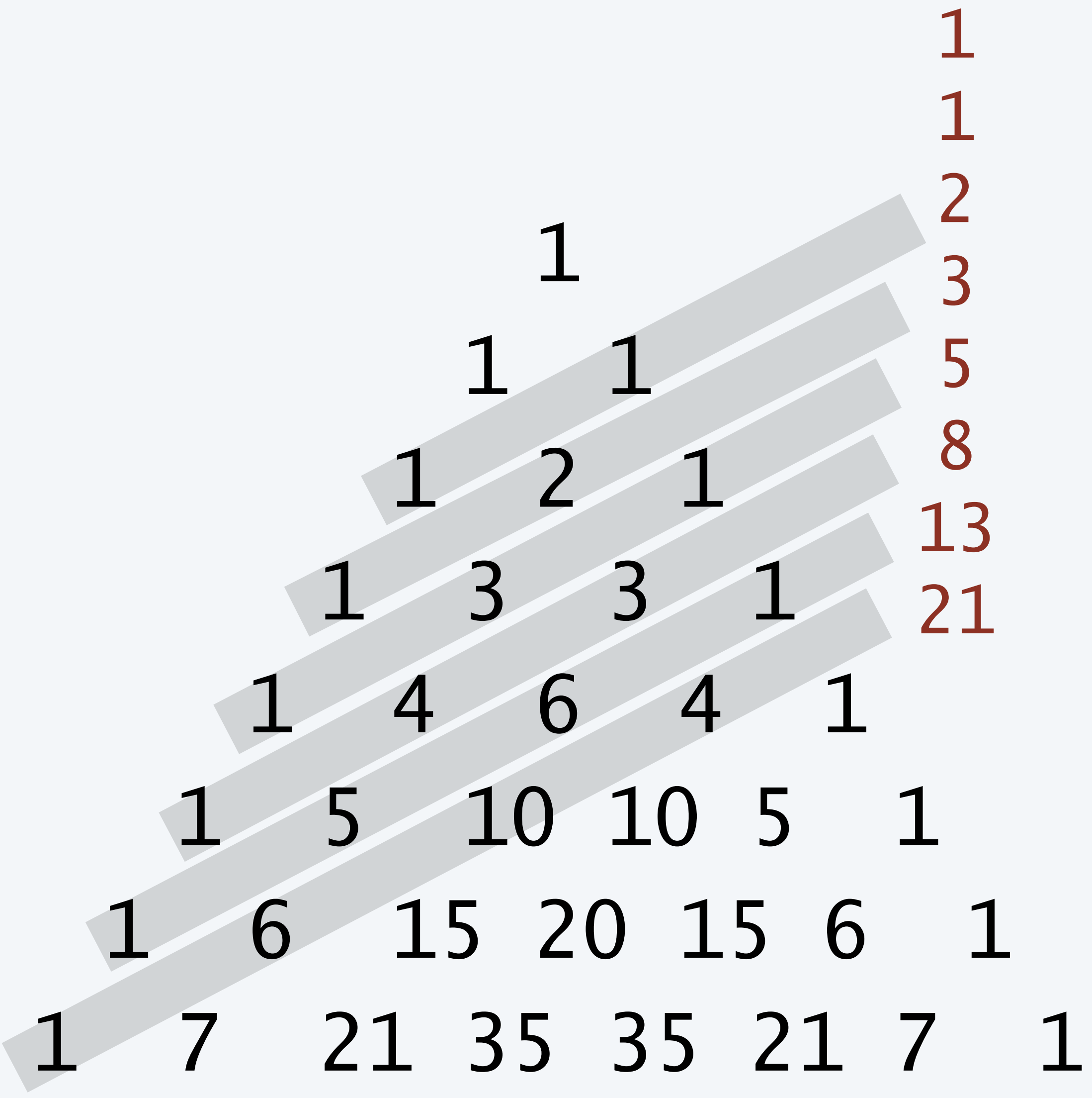
Fibonacci numbers and the golden ratio in the wild











Computing Fibonacci numbers

Q. [Curious individual.] What is the exact value of F_{60} ?

A. [Novice programmer.] Just a second. I'll write a recursive program to compute it.

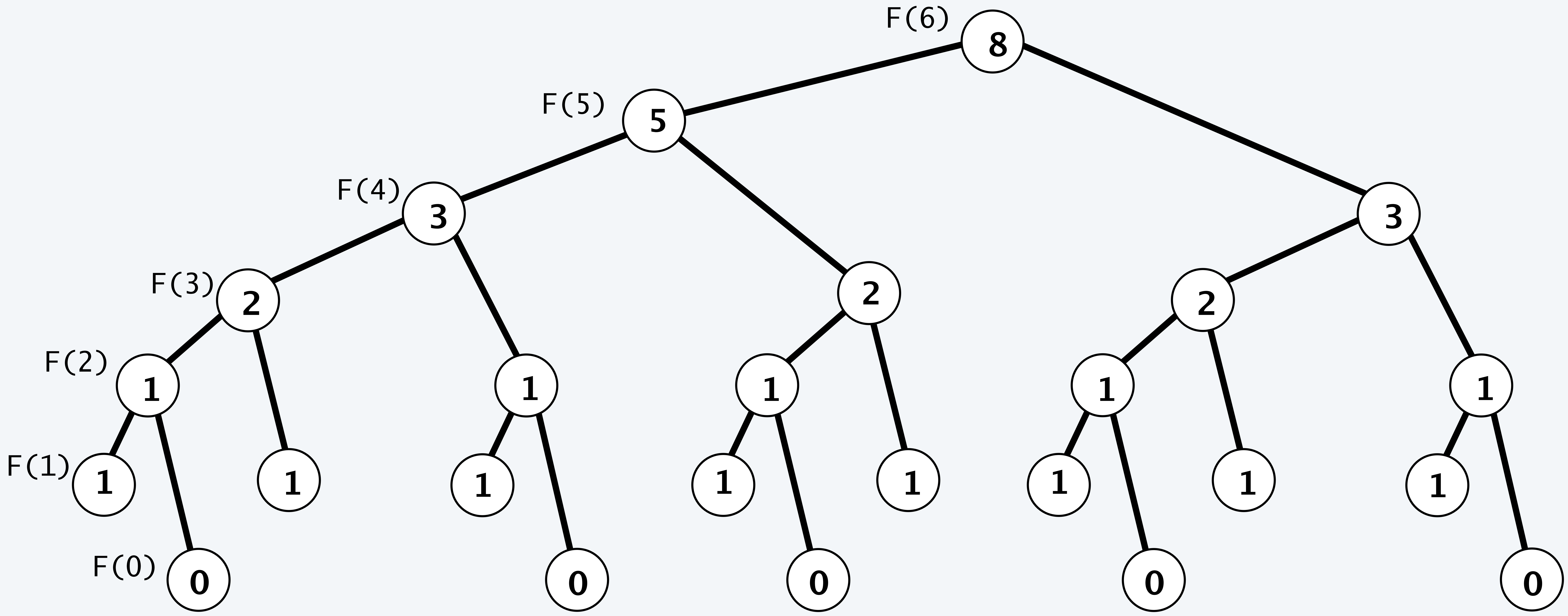
```
public class FibonacciR
{
   public static long F(int n)
   {
      if (n == 0) return 0;
      if (n == 1) return 1;
      return F(n-1) + F(n-2);
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(F(n));
   }
}
```

```
% java FibonacciR 5
5
% java FibonacciR 6
8
% java FibonacciR 10
55
% java FibonacciR 12
144
% java FibonacciR 50
12586269025
% java FibonacciR 60
```

takes a few minutes
Hmmm. Why is that?

Is something wrong with my computer?

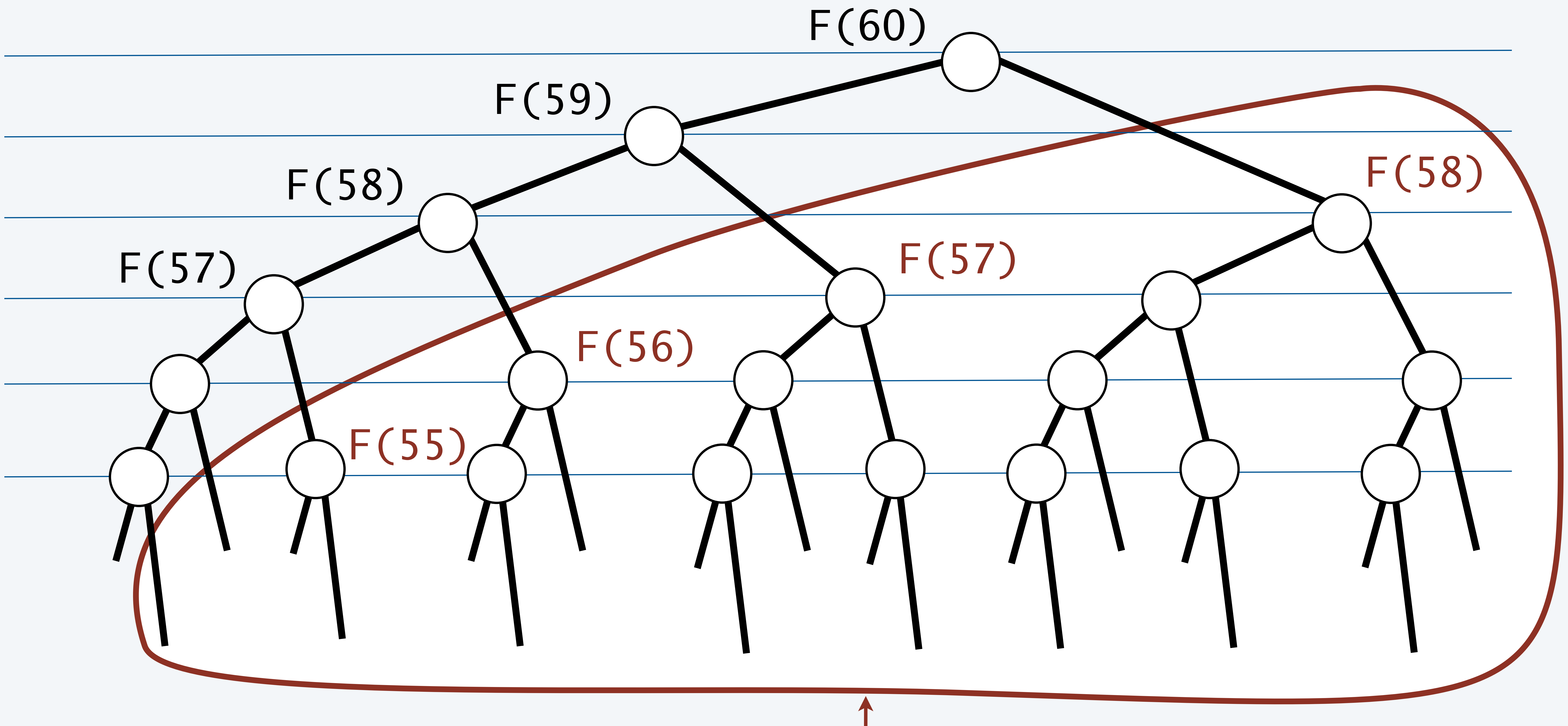
Recursive call tree for Fibonacci numbers



Exponential waste

Let C_n be the number of times F(n) is called when computing F(60).

n	C_n	
60	1	F_1
59	1	F_2
58	2	F_3
57	3	F_4
56	5	F_5
55	8	F_6
	…	…
0	$>2.5 \times 10^{12}$	F_{61}

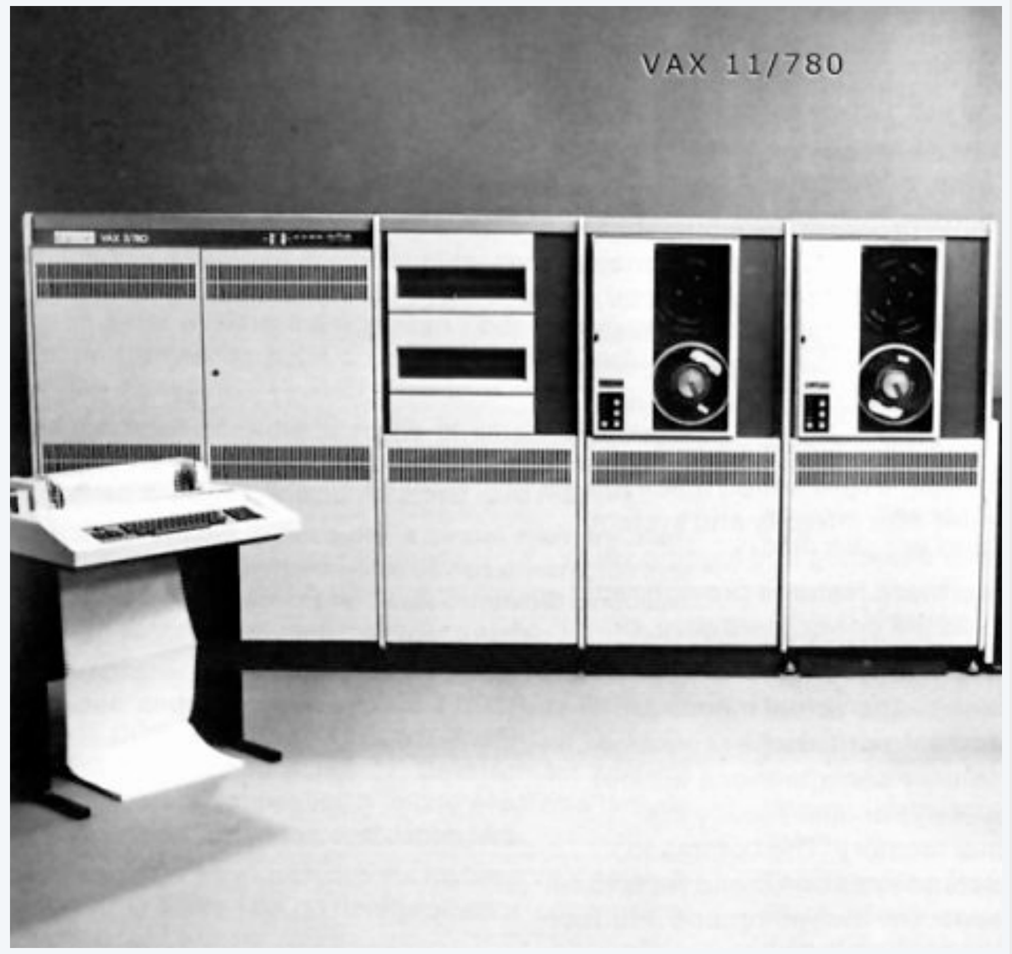


Exponentially wasteful to recompute all these values.
(trillions of calls on F(0), not to mention calls on F(1), F(2),...)

Exponential waste dwarfs progress in technology

If you engage in exponential waste, you *will not* be able to solve a large problem.

1970s



VAX 11/780

n	time to compute F_n
30	minutes
40	hours
50	weeks
60	years
70	centuries
80	millenia

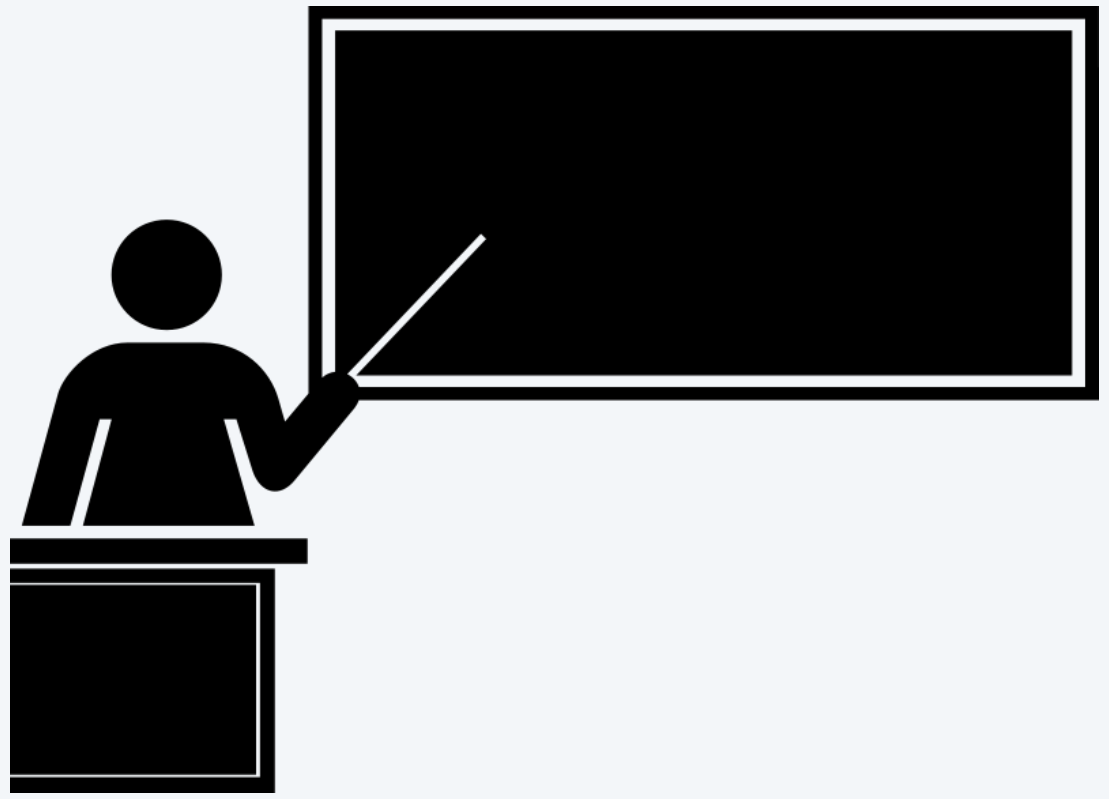
2010s: 10,000+ times faster


Macbook Air

n	time to compute F_n
50	minutes
60	hours
70	weeks
80	years
90	centuries
100	millenia

1970s: "That program won't compute F_{60} before you graduate! "

2010s: "That program won't compute F_{80} before you graduate! "



Avoiding exponential waste

Memoization

- Maintain an array memo[] to remember all computed values.
- If value known, just return it.
- Otherwise, compute it, remember it, and then return it.

```
public class FibonacciM
{
   static long[] memo = new long[100];
   public static long F(int n)
   {
      if (n == 0) return 0;
      if (n == 1) return 1;
      if (memo[n] == 0)
         memo[n] = F(n-1) + F(n-2);
      return memo[n];
   }
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      StdOut.println(F(n));
   }
}
```

```
% java FibonacciM 50
12586269025
% java FibonacciM 60
1548008755920
% java FibonacciM 80
23416728348467685
```

Simple example of *dynamic programming* (next).

Image sources

http://en.wikipedia.org/wiki/Fibonacci

http://www.inspirationgreen.com/fibonacci-sequence-in-nature.html

http://www.goldenmeancalipers.com/wp-content/uploads/2011/08/mona_spiral-1000x570.jpg

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File:Parthenon-uncorrected.jpg

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7. Recursion

- Foundations
- A classic example
- Recursive graphics
- Avoiding exponential waste
- **Dynamic programming**

An alternative to recursion that avoids recomputation

Dynamic programming.

- Build computation from the "*bottom up*".
- Solve small subproblems *and save solutions*.
- Use those solutions to build bigger solutions.

Richard Bellman
1920-1984

Fibonacci numbers

```
public class Fibonacci
{
   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      long[] F = new long[n+1];
      F[0] = 0; F[1] = 1;
      for (int i = 2; i <= n; i++)
         F[i] = F[i-1] + F[i-2];
      StdOut.println(F[n]);
   }
}
```

```
% java Fibonacci 50
12586269025
% java Fibonacci 60
1548008755920
% java Fibonacci 80
23416728348467685
```

Key advantage over recursive solution. Each subproblem is addressed only *once*.

DP example: Longest common subsequence

Def. A *subsequence* of a string s is any string formed by deleting characters from s.

Ex 1.

s = ggcaccacg

cac	ggcaccacg
gcaacg	ggcaccacg
ggcaacg	ggcaccacg
ggcacacg	ggcaccacg
...	

[2^n subsequences in a string of length n]

Ex 2.

t = acggcggatacg

gacg	acggcggatacg
ggggg	acggcggatacg
cggcgg	acggcggatacg
ggcaacg	acggcggatacg
ggggaacg	acggcggatacg
...	

longest common subsequence

Def. The *LCS* of s and t is the longest string that is a subsequence of both.

Goal. Efficient algorithm to compute the LCS and/or its length ← numerous scientific applications

Longest common subsequence

Goal. Efficient algorithm to compute the *length* of the LCS of two strings s and t.

Approach. Keep track of the length of the LCS of `s[i..M)` and `t[j..N)` in `opt[i, j]`

`s = ggcaccacg`

`t = acggcggatacg`

Three cases:

- i = M or j = N
  `opt[i][j] = 0`
- `s[i] = t[j]`
  `opt[i][j] = opt[i+1, j+1] + 1`
- otherwise
  `opt[i][j] = max(opt[i, j+1], opt[i+1][j])`

Ex: i = 6, j = 7

```
s[6..9)  = acg
t[7..12) = atacg
LCS(cg, tacg) = cg
LCS(acg, atacg) = acg
```

Ex: i = 6, j = 4

```
s[6..9)  = acg
t[4..12) = cggatacg
LCS(acg, ggatacg) = acg
LCS(cg, cggatacg) = cg
LCS(acg, cggatacg) = acg
```

LCS example

String t, indexed by j

String s, indexed by i

		0	1	2	3	4	5	6	7	8	9	10	11	12
		a	c	g	g	c	g	g	a	t	a	c	g	
0	g	?	?	?	?	?	?	?	?	?	?	?	?	0
1	g	?	?	?	?	?	?	?	?	?	?	?	?	0
2	c	?	?	?	?	?	?	?	?	?	?	?	?	0
3	a	?	?	?	?	?	?	?	?	?	?	?	?	0
4	c	?	?	?	?	?	?	?	?	?	?	?	?	0
5	c	?	?	?	?	?	?	?	?	?	?	?	?	0
6	a	?	?	?	?	?	?	?	?	?	?	?	?	0
7	c	?	?	?	?	?	?	?	?	?	?	?	?	0
8	g	?	?	?	?	?	?	?	?	?	?	?	?	0
9		0	0	0	0	0	0	0	0	0	0	0	0	0

First case:

- i = M or j = N
  `opt[i][j] = 0`

LCS example

String t, indexed by j

String s, indexed by i

		0	1	2	3	4	5	6	7	8	9	10	11	12
		a	c	g	g	c	g	g	a	t	a	c	g	
0	g	?	?	?	?	?	?	?	?	?	?	?	?	0
1	g	?	?	?	?	?	?	?	?	?	?	?	?	0
2	c	?	?	?	?	?	?	?	?	?	?	?	?	0
3	a	?	?	?	?	?	?	?	?	?	?	?	?	0
4	c	?	?	?	?	?	?	?	?	?	?	?	?	0
5	c	?	?	?	?	?	?	?	?	?	?	?	?	0
6	a	?	?	?	?	?	?	?	?	?	?	?	?	0
7	c	?	?	?	?	?	?	?	?	?	?	?	?	0
8	g	?	?	?	?	?	?	?	?	?	?	?	1	0
9		0	0	0	0	0	0	0	0	0	0	0	0	0

`opt[i][j] = opt[i+1, j+1] + 1`

LCS example

String t, indexed by j

String s, indexed by i

i	s	0	1	2	3	4	5	6	7	8	9	10	11	12
		a	c	g	g	c	g	g	a	t	a	c	g	
0	g	7	7	7	6	6	6	5	4	3	3	2	1	0
1	g	6	6	6	6	5	5	5	4	3	3	2	1	0
2	c	6	5	5	5	5	4	4	4	3	3	2	1	0
3	a	6	5	4	4	4	4	4	4	3	3	2	1	0
4	c	5	5	4	4	4	3	3	3	3	3	2	1	0
5	c	4	4	4	4	4	3	3	3	3	3	2	1	0
6	a	3	3	3	3	3	3	3	3	3	3	2	1	0
7	c	2	2	2	2	2	2	2	2	2	2	2	1	0
8	g	1	1	1	1	1	1	1	1	1	1	1	1	0
9		0	0	0	0	0	0	0	0	0	0	0	0	0

`opt[i][j] = max(opt[i, j+1], opt[i+1][j])`

`opt[i][j] = opt[i+1, j+1] + 1`

LCS length implementation

```
public class LCS
{
    public static void main(String[] args)
    {
        String s = args[0];
        String t = args[1];
        int M = s.length();
        int N = t.length();

        int[][] opt = new int[M+1][N+1];

        for (int i = M-1; i >= 0; i--)
            for (int j = N-1; j >= 0; j--)
                if (s.charAt(i) == t.charAt(j))
                    opt[i][j] = opt[i+1][j+1] + 1;
                else
                    opt[i][j] = Math.max(opt[i+1][j], opt[i][j+1]);

        System.out.println(opt[0][0]);
    }
}
```

```
% java LCS ggcaccacg acggcggatacg
7
```

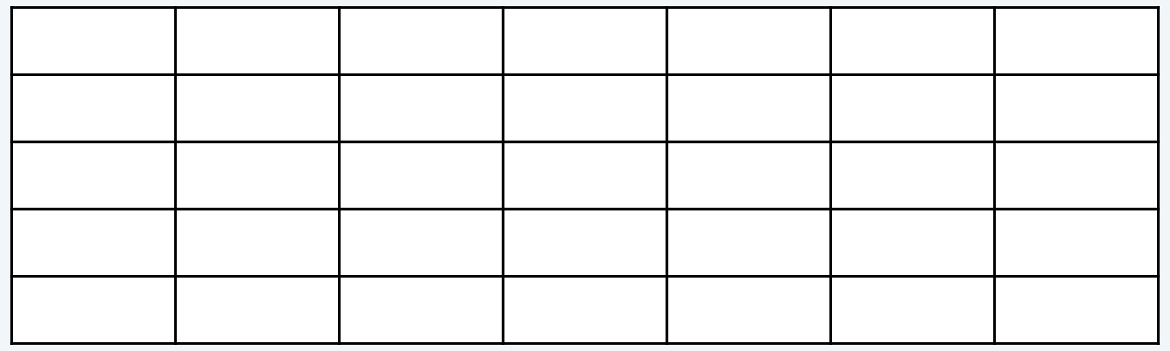
Exercise. Add code to print LCS itself (see `LCS.java` on booksite for solution).

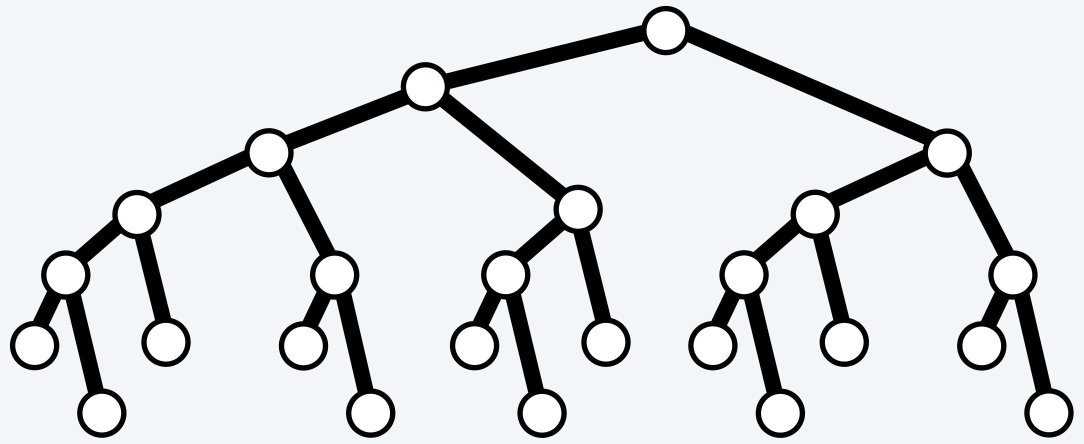
Dynamic programming and recursion

Broadly useful approaches to solving problems by combining solutions to smaller subproblems.

Why learn DP and recursion?

- Represent a new mode of thinking.
- Provide powerful programming paradigms.
- Give insight into the nature of computation.
- Successfully used for decades.





	recursion	*dynamic programming*
advantages	Decomposition often obvious. Easy to reason about correctness.	Avoids exponential waste. Often simpler than memoization.
pitfalls	Potential for exponential waste. Decomposition may not be simple.	Uses significant space. Not suited for real-valued arguments. Challenging to determine order of computation

Image sources

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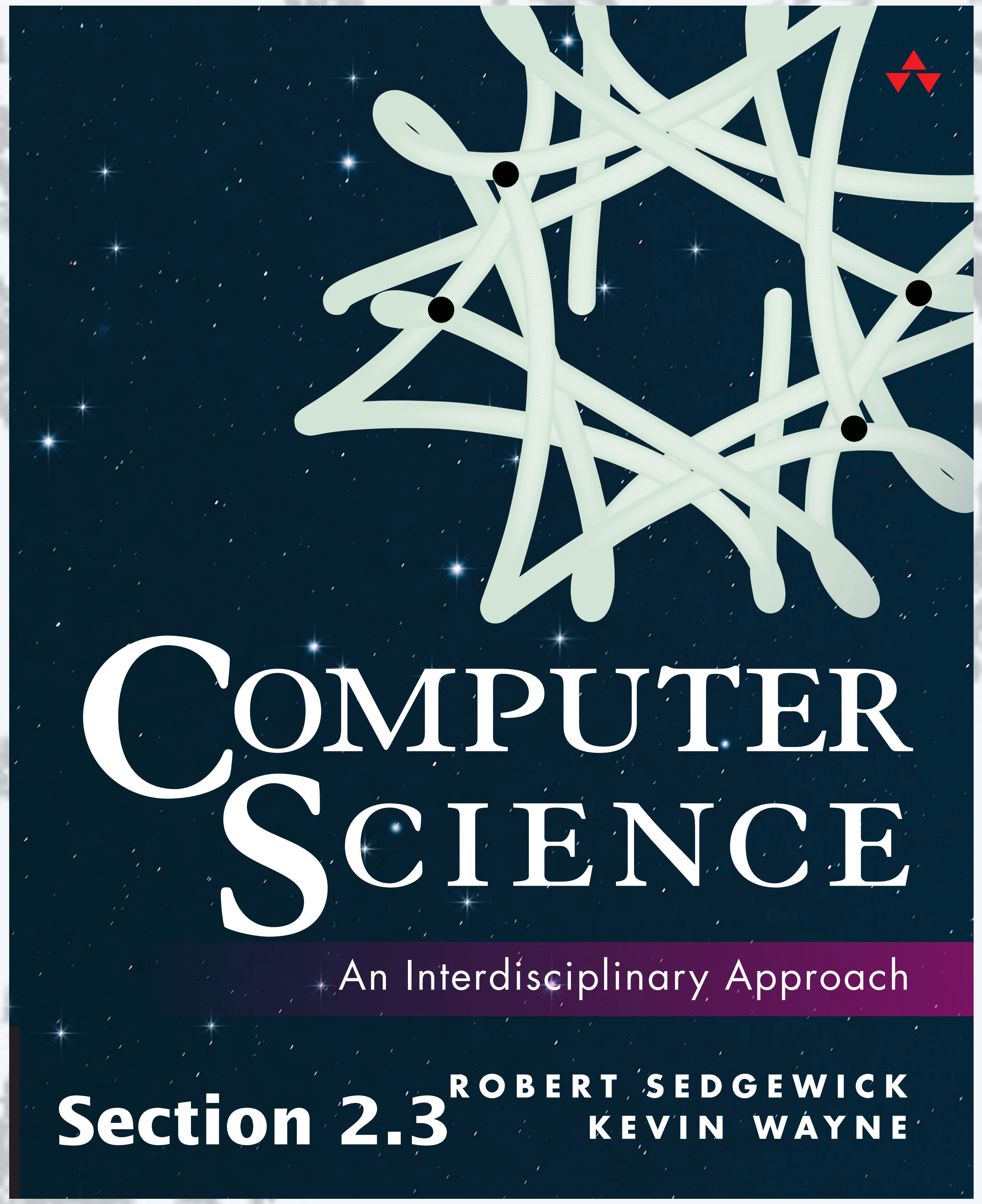
COMPUTER SCIENCE

SEDGEWICK / WAYNE

PART I: PROGRAMMING IN JAVA



http://introcs.cs.princeton.edu

6. Recursion

Coin Changing

Acknowledgements:
Virginia, Princeton, Penn, Washington Post

Coin Changing

Given access to an unlimited number of pennies, nickels dimes, and quarters, give an algorithm which gives change for a target value x using the <u>fewest</u> number of coins.

Coin Changing: Greedy Algorithm

Given: target value x, list of coins $C = [c_1, \ldots, c_n]$
(in this case $C = [1, 5, 10, 25]$)

Repeatedly select the largest coin less than the remaining target value:

while $x > 0$:
    let $c = \max(c_i \in \{c_1, \ldots, c_n\} \mid c_i \leq x)$
    add c to list L
    $x = x - c$
output L

Example of a **greedy algorithm**:
always choose the "optimal" choice

How to make 90 cents?

Coin Changing: Greedy Solution

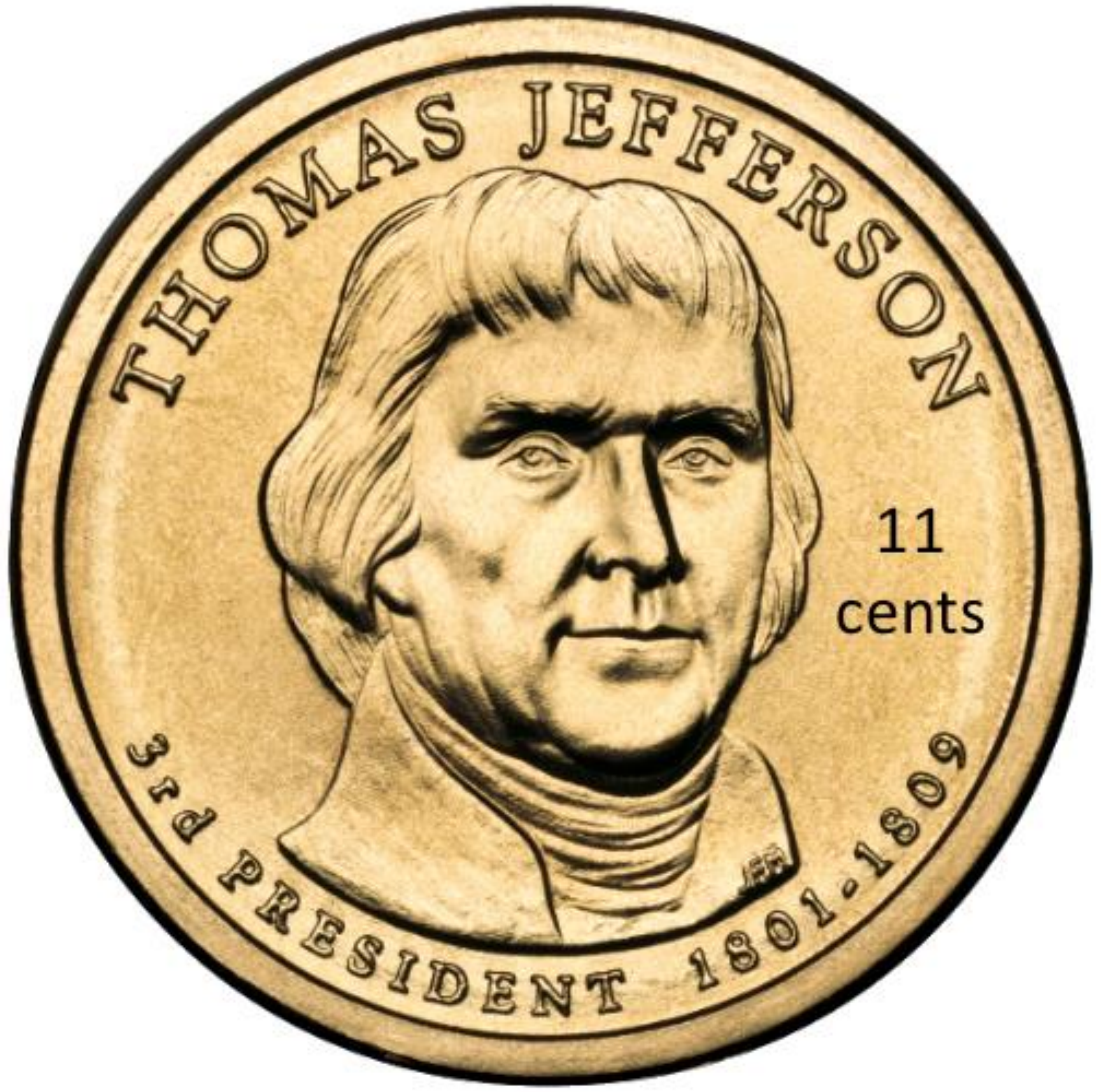
90 cents

When can we use the greedy solution?

Optimal!

Coin Changing: Greedy Solution

Suppose we added a new coin worth 11 cents. In conjunction with pennies, nickels, dimes, and quarters, find the minimum number of coins needed to give 90 cents of change.



Coin Changing: Greedy Solution

90 cents

Stamps: greedy != optimal

- Denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500
- How to make 140?
  - Optimal Solution? Greedy Solution?

Cashier's Algorithm

- Repeatedly:
  - Add coin of the largest value that does not take us past the amount to be paid
- This is a greedy algorithm
- Assume we have coins worth:
  - 100¢, 25¢, 10¢, 5¢, 1¢
- Is this greedy algorithm optimal (i.e., does it use the fewest number of coins)?

Properties of any optimal solution (for U.S. coin denominations)

Property. Number of pennies ≤ 4.

Pf. Replace 5 pennies with 1 nickel.

Property. Number of nickels ≤ 1.

Property. Number of quarters ≤ 3.

Property. Number of nickels + number of dimes ≤ 2.

Pf.

- Recall: ≤ 1 nickel.
- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dimes and 1 nickel with 1 quarter.

dollars (100¢) | quarters (25¢) | dimes (10¢) | nickels (5¢) | pennies (1¢)

Optimality of cashier's algorithm (for U.S. coin denominations)

Theorem. Cashier's algorithm is optimal for U.S. coins { 1, 5, 10, 25, 100 }.

Pf. [by induction on amount to be paid x]

- Consider optimal way to change $c_k \leq x < c_{k+1}$: greedy takes coin k.
- We claim that any optimal solution must take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by cashier's algorithm. ▪

k	c_k	all optimal solutions must satisfy	max value of coin denominations $c_1, c_2, \ldots, c_{k-1}$ in any optimal solution	
1	1	$P \leq 4$	–	
2	5	$N \leq 1$	4	← 4 pennies
3	10	$N + D \leq 2$	4 + 5 = 9	← 4P + 1 nickel
4	25	$Q \leq 3$	20 + 4 = 24	← 4P + 2 dimes
5	100	*no limit*	75 + 24 = 99	← 3Q + 4P + 2D

General Coin Changing Algorithm

- So, the greedy cashier's algorithm works…
- …if we assume we have coins worth:
  - 100¢, 25¢, 10¢, 5¢, 1¢
- But as in the postage stamp example, with different coin values, a greedy algorithm may **not** be optimal
- Is there an algorithm that works, for any set of coin/stamp values?
  - Yes, as we will see next!

General Coin Changing Algorithm: Recursion

- We can reduce the problem recursively by choosing the first coin, and solving for the amount that is left
- For a target value x (e.g., x = 99¢), and the coin set with denominations $\{d_1, d_2, \ldots, d_n\}$
- Choose the best solution from:
  - One d_1 coin plus the best solution for $(x - d_1)$
  - One d_2 coin plus the best solution for $(x - d_2)$
  - …
  - One d_n coin plus the best solution for $(x - d_n)$
- If $d_i > x$, we say that it takes ∞ coins to make change, to indicate that it's impossible
- However… this algorithm is inefficient, because **overlapping subproblems are solved repeatedly**

General Coin Changing Algorithm – Dynamic Programming

- Key Idea: Solve the problem first for one cent, then two cents, then three cents, etc., up to the desired amount
- *Save each answer along the way !*
- For each new amount N, compute all the possible pairs of previous answers which sum to N
- For example, to find the solution for 13¢,
- First, solve for all of 1¢, 2¢, 3¢, ..., 12¢
- Next, choose the best solution among:
- Solution for 1¢ + solution for 12¢
- Solution for 2¢ + solution for 11¢
- Solution for 3¢ + solution for 10¢
- Solution for 4¢ + solution for 9¢
- Solution for 5¢ + solution for 8¢
- Solution for 6¢ + solution for 7¢

- This is great! How to manage this process in general?

Dynamic Programming (DP)

- Powerful technique for optimization problems with
  - Optimal sub-structure: optimal solution to a larger problem contains the optimal solutions to smaller ones
  - Overlapping sub-problems
- General process for developing a DP solution
  - Define sub-problems
  - Identify recurrence relations among sub-problems
  - Find a good order to solve the sub-problems, save their solutions, and finally solve the original problem
    - Top-down recursion with memoization: larger problems → related smaller problems
    - Bottom-up iteration: smaller problems → larger problems

skip coin i — use coin i

$$c(i,j) = \begin{cases} 0 & \text{if } j = 0 \\ \frac{j}{d_1} & \text{if } i = 1 \\ \infty & \text{if } j < 0 \\ \min(c(i-1,j),\ 1 + c(i, j-d_i)) & \text{otherwise} \end{cases}$$

Given a new coin *i*, what's the fewest coins required to make *j* in change?

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senum=1	0	1	2	3	4	5	6	7
seon=2								
shum=4								
limnah=7								

c[i,j] = min. number of "coins" to make j change with coins 1..i

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2								
shum=4								
limnah=7								

c[i,j] = min. number of coins to make j change with coins 1..i.

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2	0	1	???					
shum=4								
limnah=7								

How does one compute c[2,2]?

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2	0	1	1					
shum=4		+1						
limnah=7								

How does one compute c[2,2]?

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2	0	1	1	2				
shum=4								
limnah=7								

+1

How does one compute c[2,3]?

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2	0	1	1	2	2			
shum=4								
limnah=7								

+1

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2	0	1	1	2	2	3		
shum=4					+1			
limnah=7								

Making Change

j →

i ↓

Amount	0	1	2	3	4	5	6	7
senine=1	0	1	2	3	4	5	6	7
seon=2	0	1	1	2	2	3	3	4
shum=4	0	1	1	2	1	2	2	3
limnah=7	0	1	1	2	1	2	2	1